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Ergodicity for the Dissipative Boussinesq Equations with Random Forcing

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We study the stationary measure for the two-dimensional Boussinesq equation with random forcing. We prove the ergodicity for the two-dimensional stochastically forced Boussinesq equation. We also study the Galerkin truncations of the three-dimensional Boussinesq equations under degenerate stochastic forcing. We follow closely the previous results on the stochastically forced Navier–Stokes equations.

KEY WORDS: Boussinesq equations; stochastic equations; ergodicity; invariant measure.

1. INTRODUCTION

We are interested in the following n-dimensional stochastically forced dissipative Boussinesq equations with $n=2$ or 3:

$$\begin{aligned} &\frac{\partial u}{\partial t}+(u\cdot\nabla)u+\nabla p-\nu\Delta u+\sigma\theta\vec{e}_n=\frac{\partial W_u(x,t)}{\partial t},\\ &\nabla\cdot u=0,\\ &\frac{\partial\theta}{\partial t}+(u\cdot\nabla)\theta-\kappa\Delta\theta=\frac{\partial W_\theta(x,t)}{\partial t}, \end{aligned}\tag{1.1}$$

where u is the fluid velocity vector field, p is the scalar pressure, θ is the scalar temperature, ν is the positive normalised fluid viscosity or Prandtl number, κ is the positive thermal diffusivity or Lewis number, σ is the Rayleigh number. $\sigma\theta\vec{e}_n$ can be interpreted as the gravitational force,

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i.e., $\vec{e}_n$ is the unit vertical direction to the earth and W_u, W_θ are the white noises. Boussinesq system in itself has an important physical meaning. Furthermore, two-dimensional Boussinesq system has the similarity with the three-dimensional axisymmetric flows. The global in time existence of the strong solution for two-dimensional deterministic dissipative Boussinesq system is standard and well known (for the recent progress on the dissipative Boussinesq type equations, see refs. 1 and 2). The questions we are interested in are the statistical properties of the stochastic system. Our first main result is that if all the determining modes are forced then the two-dimensional stochastically forced Boussinesq equation possesses a unique stationary measure. The ergodicity of the two-dimensional stochastically forced Navier–Stokes equation has been intensively studied by many authors (e.g. see refs. 3–10 and references therein). Moreover, the ergodicity for the more general dissipative equations including stochastic Ginzburg–Landau equations was studied in ref. 11. We follow the strategy of ref. 6 and 11 closely. For the proof of the first main result, we take W_u and W_θ as the following simple form. For $N>0$, let

$$W_u(x,t)=\sum_{|k|\leqslant N}\sigma_k w_k(t,\omega)e_k(x)$$

and

$$W_\theta(x,t)=\sum_{|k|\leqslant N}\tilde{\sigma}_k w_k(t,\omega)\tilde{e}_k(x),$$

where w_k's are standard complex valued Wiener process satisfying $w_{-k}(t)=w_k(t)$ and $\sigma_k,\tilde{\sigma}_k\in\mathbb{C}$ with $|\sigma_k|>0$, are the amplitudes of the forcing,

$$\left\{e_k(x)=\begin{pmatrix}-ik_2\\ ik_1\end{pmatrix}(e^{ik\cdot x}/|k|),\ k\in\mathbb{Z}^2\right\}$$

are the basis in the space of L^2 divergence-free, mean zero vector fields on $\mathbb{T}^2$, the two-dimensional torus and $\{\tilde{e}_k(x)=e^{ik\cdot x},\ k\in\mathbb{Z}^2\}$ are the basis in the space of $\tilde{L}^2$ scalar fields on $\mathbb{T}^2$(we denote $\tilde{L}^2$ for the discrimination of L^2 vector field). We denote $L^2\oplus\tilde{L}^2$ by $\mathbb{L}^2$. Define $B(u,v)=-P_{\mathrm{div}}(u\cdot\nabla)v$, $\Lambda^2u=-P_{\mathrm{div}}\Delta u$, where P_{div} is the L^2 projection operator onto the space of divergence-free vector fields. Let $\sigma_{\max}^2=\max\{|\sigma_k|^2 : |k|\leqslant N\}$, $\tilde{\sigma}_{\max}^2=\max\{|\tilde{\sigma}_k|^2 : |k|\leqslant N\}$, $\mathcal{E}_l^u=\sum_{|k|\leqslant N}|k|^{2l}|\sigma_k|^2$ and $\mathcal{E}_l^\theta=\sum_{|k|\leqslant N}|k|^{2l}|\tilde{\sigma}_k|^2$. Writing $u(x)=\sum_k u_k e_k(x)$ and $\theta(x)=\sum_k\theta_k\tilde{e}_k(x)$, we will define $H^\alpha=\{u=(u_k),\ u_0=0,\ \sum_k|k|^{2\alpha}|u_k|^2<\infty\}$ and $\tilde{H}^\alpha=\{\theta=(\theta_k),\ \theta_0=0,\ \sum_k|k|^{2\alpha}|\theta_k|^2<\infty\}$. In the following, we also denote $\tilde{H}^\alpha$ by H^α for simplicity. We will work on the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}, \chi_t)$. We associate Ω with the canonical space generated by all $d\omega_k(t)$. $\mathcal{F}$ and $\mathcal{F}_t$ are, respectively, the associated global σ-algebra and filtration generated by $W(t)$. Lastly, χ_t is the shift on Ω defined by $\chi_t\, d\omega_k(s) = d\omega_k(s+t)$. Notice that χ_t is an ergodic group of measure-preserving transformations with respect to $\mathbb{P}$. Expectations with respect to $\mathbb{P}$ will be denoted by $\mathbb{E}$. Projecting (1.1) onto the divergence-free vector field, we obtain the following system of Ito stochastic equation:

$$\begin{aligned} &du + \nu\Lambda^2 u(x, t)dt = B(u, u)dt + \sigma a(\theta)dt + dW_u(x, t), \\ &d\theta - \kappa\Delta\theta(x, t)dt = -(u\cdot\nabla)\theta dt + dW_\theta(x, t), \end{aligned} \tag{1.2}$$

where $a(x) = -P_{\text{div}}(x\vec{e}_n)$. (1.2) generates a continuous Markovian stochastic semi-flow on $\mathbb{L}^2 = L^2 \times \tilde{L}^2$ defined by $\phi^{\omega}_{s,t}(u_0, \theta_0) = (u, \theta)(t, \omega; s, (u_0, \theta_0))$. When $s = 0$, we write ϕ^{ω}_t.

Theorem 1.1. There exist some absolute constant $\hat{C}$ and C_1 such that if $N^2 \geqslant \hat{C}(\mathcal{E}^u_0 + C_1\mathcal{E}^\theta_0)$, then (1.2) has a unique stationary measure on $\mathbb{L}^2$.

The strategy of the proof of Theorem 1.1 is as follows. The existence of the stationary measure is rather standard(see ref. 4). Hence in this paper, we are only interested in the uniqueness of the stationary measure. The uniqueness of the stationary measure of two-dimensional Navier–Stokes equations is strongly studied by many authors. Our strategy for the proof of Theorem 1.1 is to reduce the dynamics of the Boussinesq equation to the dynamics of a finite dimensional set of low modes with memory. The reduced dynamics are no longer Markovian but rather Gibbsian. The finite dimensional Gibbsian dynamics have a nondegenerate noise, and have a unique stationary measure if the memory is short ranged. If μ is any given stationary measure on L^2, then it can be extended to a measure μ_p on the path space $C((-\infty, 0], \mathbb{L}^2)$. Let $A = \{(u(s), \theta(s)) \in C((-\infty, 0], \mathbb{L}^2), u(t_i) \in A_i, i = 0, ..., n\}$, where A_i are Borel sets of $\mathbb{L}^2$ for some $t_0, ..., t_n$. We let $B = \{((u, \theta), \omega), (u, \theta) \in A_0, \phi^{\omega}_{t_0,t_i}(u, \theta) \in A_i, i = 1, ..., n\}$. We will define $\mu_p(A) = (\mathbb{P}\times\mu)(B)$. Symbolically, if $(u(\cdot), \theta(\cdot)) \in C((-\infty, 0], \mathbb{L}^2)$, then $(\psi^{\omega}_t(u, \theta))(s) = \phi^{\omega}_s(u, \theta)(0)$ for some $s \in [0, t]$ and $(\psi^{\omega}_t(u, \theta))(s) = (u(s), \theta(s))$ for $s \leqslant 0$. If μ is invariant then μ_p is invariant in the sense that

$$\int_{C((-\infty,0],\mathbb{L}^2)} F((u, \theta))d\mu_p(u, \theta) = \mathbb{E}\int_{C((-\infty,0],\mathbb{L}^2)} F(\chi_t\psi^{\omega}_t(u,\theta))d\mu_p(u,\theta)$$

for all bounded functions on $C((-\infty, 0], \mathbb{L}^2)$ and $t \geqslant 0$.

For our second main result, we consider the finite dimensional Galerkin approximations of the three-dimensional dissipative Boussinesq equations with degenerate stochastical forcing in the domain $\mathbb{T}^3$, with periodic boundary conditions, where W_u, W_θ are Brownian motions with some simplifying assumptions stated later. We write

$$u(t,x) = \sum_{k\in\mathbb{Z}^3} u_k(t)e^{ik\cdot x},$$
$$\theta(t,x) = \sum_{k\in\mathbb{Z}^3} \theta_k(t)e^{ik\cdot x},$$

where $u_{-k} = \bar{u}_k$ and $\theta_{-k} = \bar{\theta}_k$. Then as usual we project our equation on the space of divergence-free vector fields and finally we take the finite dimensional Galerkin truncation. We also set

$$dW_u = \sum q_{uk}\, d\beta_{ut}^k\, e^{ik\cdot x} \quad \text{and} \quad dW_\theta = \sum q_{\theta k}\, d\beta_{\theta t}^k\, e^{ik\cdot x}.$$

Hence we obtain the following finite-dimensional system of stochastic differential equations:

$$\begin{aligned} du_k &= [-\nu|k|^2u_k - i\sum(k\cdot u_h)(u_l - \frac{k\cdot u_l}{|k|^2}k) \\ &\quad -\sigma(\tilde{\theta}_k - \frac{k\cdot\tilde{\theta}_k}{|k|^2}k)]dt + q_{uk}\, d\beta_{ut}^k, \\ d\theta_k &= [-\kappa|k|^2\theta_k - i\sum(k\cdot u_h)\theta_l]dt + q_{\theta k}\, d\beta_{\theta t}^k, \end{aligned}$$

where $\tilde{\theta}_k = (0,0,\theta_k)^T$, N is a fixed threshold, and sum is over $h,\, l\in\mathcal{K}_N$ and $h+l=k$. We define

$$\mathcal{K}_N = \{k\in\mathbb{Z}^3 \mid |k|_\infty \leqslant N,\ k\neq(0,\ 0,\ 0)\},$$

where we exclude $(0,0,0)$ because in this part we are interested in the case under no mean stochastic forcing. For simplicity, we assume that the noise takes values in the space of divergence-free vector fields and covariance is diagonal in the Fourier space. Now we are ready to state our second theorem.

Theorem 1.2. Let $\mathcal{K}$ be the set of modes forced. Assume that the noise satisfies the above assumptions and the set $\mathcal{K}$ contains the three indices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Then the finite dimensional system admits a unique invariant measure.

The existence of an invariant measure can be proved by standard compactness argument. For the proof of the uniqueness, we reduced it to two parts. First, we prove the transition probability densities are regular, i.e., the diffusion operator is hypoelliptic. Then we prove the associated Markov process is irreducible in the sense starting from any initial position, the dynamics enters any neighborhood of the origin infinitely often(see refs. 12 and 13). Once we have these two properties, we can follow some arguments in ref. 4 to prove Theorem 1.2. Recently, many authors have intensively studied above techniques for the ergodicity of the solutions of the stochastic equations(see refs. 5, 14 and 15 and references therein). For this part, we follow closely refs. 5 and 15. In the following, C will be used as a generic constant.

2. ENERGY ESTIMATES

In this section, we derive energy and enstrophy estimates using similar method in refs. 6 and 9.

We fix a positive integer M and consider the Galerkin approximations defined by $u^{(M)}(t) = \sum_{|k|\leqslant M} u_k^{(M)}(t)e_k$ and $\theta^{(M)}(t) = \sum_{|k|\leqslant M} \theta_k^{(M)}(t)\tilde{e}_k$. $u^{(M)}(t)$ and $\theta^{(M)}(t)$ satisfy an equation of exactly the same form as the full solution except the nonlinearity has been projected to the terms of order less than or equal to M. We let $\mathcal{E}_l^{uM} = \sum_{|k|\leqslant M} |k|^{2l}|\sigma_k|^2$ and $\mathcal{E}_l^{\theta M} = \sum_{|k|\leqslant M} |k|^{2l}|\tilde{\sigma}_k|^2$. Since our estimates are independent of the order of approximations M, we sometimes neglect the superscript M. Applying Ito's formula to the maps $\{u_k\}\to(\sum|u_k|^2)^p$ and $\{\theta_k\}\to(\sum|\theta_k|^2)^p$, we have the followings for the energy moments:

$$\begin{aligned}
d|u(t)|_{L^2}^{2p} &= 2p|u(t)|_{L^2}^{2(p-1)}[-\nu|\Lambda u(t)|_{L^2}^2\,dt + \sigma\langle a(\theta(t)), u(t)\rangle_{L^2}\,dt \\
&\quad + \langle u(t), dW_u\rangle_{L^2}] + 2p(p-1)|u(t)|_{L^2}^{2(p-2)}\Big(\sum|u_k(t)|^2|\sigma_k|^2\Big)\,dt \\
&\quad + p|u(t)|_{L^2}^{2(p-1)}\mathcal{E}_0^{uM}\,dt
\end{aligned}$$

and

$$\begin{aligned}
d|\theta(t)|_{L^2}^{2p} &= 2p|\theta(t)|_{L^2}^{2(p-1)}[-\kappa|\nabla\theta(t)|_{L^2}^2\,dt + \langle\theta(t), dW_\theta\rangle_{L^2}] \\
&\quad + 2p(p-1)|\theta(t)|_{L^2}^{2(p-2)}\Big(\sum|\theta_k(t)|^2|\tilde{\sigma}_k|^2\Big)\,dt \\
&\quad + p|\theta(t)|_{L^2}^{2(p-1)}\mathcal{E}_0^{\theta M}\,dt.
\end{aligned}$$

Here $\langle\Lambda^\alpha u(t), dW_u(t)\rangle_{L^2}$ and $\langle\Lambda^\alpha\theta(t), dW_\theta(t)\rangle_{L^2}$ denote $\sum_k |k|^\alpha u_k(t)\sigma_k dw_k(t)$ and $\sum_k |k|^\alpha\theta_k(t)\tilde{\sigma}_k d\tilde{w}_k(t)$, respectively. The analysis of the energy moments of the Boussinesq equations is slightly different from that of two-dimensional Navier–Stokes equations. First we remark that

$$|a(\theta(t))|_{L^2} \leqslant C|\theta(t)|_{L^2}.$$

For a fixed $H > 0$, we denote the stopping time $T = \inf\{t \geqslant 0 : |u(t)|_{L^2}^2 \geqslant H^2 \text{ or } |\theta(t)|_{L^2}^2 \geqslant H^2\}$. Denoting by M_t^u and M_t^θ the local martingale term, we denote the stopped martingales by M_t^{uT} and $M_t^{\theta T}$. The quadratic variation of M_t^{uT} and $M_t^{\theta T}$ are finite as follows:

$$[M^{uT}, M^{uT}]_t \leqslant 2p(\sigma_{\max})^2 \int_0^t |u(s \wedge T)|_{L^2}^{2p}\, ds \leqslant 2p(\sigma_{\max})^2 H^{2p} t < \infty$$

and

$$[M^{\theta T}, M^{\theta T}]_t \leqslant 2p(\tilde{\sigma}_{\max})^2 \int_0^t |\theta(s \wedge T)|_{L^2}^{2p}\, ds \leqslant 2p(\tilde{\sigma}_{\max})^2 H^{2p} t < \infty.$$

Since $\mathbb{E}[M^{uT}, M^{uT}]$, $\mathbb{E}[M^{\theta T}, M^{\theta T}] < \infty$, we have $\mathbb{E}M_t^{uT} = \mathbb{E}M_t^{\theta T} = 0$. Using the optional stopping time lemma and the fact that $M_{t\wedge T}^u = M_{t\wedge T}^{uT}$ and $M_{t\wedge T}^\theta = M_{t\wedge T}^{\theta T}$, we have

$$\begin{aligned}
&\mathbb{E}|u(t \wedge T)|_{L^2}^{2p} + 2p\nu\mathbb{E}\int_0^{t\wedge T} |u(s)|_{L^2}^{2(p-1)} |\Lambda u(s)|_{L^2}^2\, ds \\
&\quad \leqslant \mathbb{E}|u(0)|_{L^2}^{2p} + p\nu\mathbb{E}\int_0^t |u(s)|_{L^2}^{2p}\, ds + p\kappa C_1\mathbb{E}\int_0^{t\wedge T} |\theta(s)|_{L^2}^{2p}\, ds \\
&\qquad + 2p(p-1)\mathbb{E}\int_0^{t\wedge T} |u(s)|_{L^2}^{2(p-2)} \Big(\sum_k |u_k(s)|^2 |\sigma_k|^2\Big)\, ds \\
&\qquad + p\mathbb{E}\int_0^{t\wedge T} |u(s)|_{L^2}^{2(p-1)} \mathcal{E}_0^{uM}\, ds
\end{aligned}$$

and

$$\begin{aligned}
&C_1\mathbb{E}|\theta(t \wedge T)|_{L^2}^{2p} + 2p\kappa C_1\mathbb{E}\int_0^{t\wedge T} |\theta(s)|_{L^2}^{2(p-1)} |\nabla\theta(s)|_{L^2}^2\, ds \\
&\quad \leqslant C_1\mathbb{E}|\theta(0)|_{L^2}^{2p} + 2p(p-1)C_1\mathbb{E}\int_0^{t\wedge T} |\theta(s)|_{L^2}^{2(p-2)} \Big(\sum |\theta_k(s)|^2 |\tilde{\sigma}_k|^2\Big)\, ds \\
&\qquad + pC_1\mathbb{E}\int_0^{t\wedge T} |\theta(s)|_{L^2}^{2(p-1)} \mathcal{E}_0^{\theta M}\, ds.
\end{aligned}$$

Using Poincaré's inequality produces that

$$\begin{aligned}
&\mathbb{E}|u(t\wedge T)|_{L^2}^{2p}+C_1\mathbb{E}|\theta(t\wedge T)|_{L^2}^{2p}+p\nu\int_0^t \mathbb{E}|\Lambda u(s)|_{L^2}^2|u(s)|_{L^2}^{2(p-1)}\,ds\\
&\quad+p\kappa C_1\int_0^{t\wedge T}\mathbb{E}|\nabla\theta(s)|_{L^2}^2|\theta(s)|_{L^2}^{2(p-1)}\,ds\\
&\leqslant \mathbb{E}|u(0)|_{L^2}^{2p}+C_1\mathbb{E}|\theta(0)|_{L^2}^{2p}\\
&\quad+[2p(p-1)(\sigma_{\max})^2+p\mathcal{E}_0^{uM}]\mathbb{E}\int_0^{t\wedge T}|u(s)|_{L^2}^{2(p-1)}\,ds\\
&\quad+C_1[2p(p-1)(\tilde{\sigma}_{\max})^2+p\mathcal{E}_0^{\theta M}]\int_0^{t\wedge T}\mathbb{E}|\theta(s)|_{L^2}^{2(p-1)}\,ds.
\end{aligned}$$

Since $u(t)$ and $\theta(t)$ are continuous in time, $T\to\infty$ as $H\to\infty$ and $t\wedge T\to t$. By taking $M\to\infty$, we have for $p=1$,

$$\begin{aligned}
&\mathbb{E}|u(t)|_{L^2}^2+C_1\mathbb{E}|\theta(t)|_{L^2}^2+\nu\mathbb{E}\int_0^t|\Lambda u(s)|_{L^2}^2\,ds+\kappa C_1\mathbb{E}\int_0^t|\nabla\theta(s)|_{L^2}^2\,ds\\
&\leqslant \mathbb{E}|u(0)|_{L^2}^2+C_1\mathbb{E}|\theta(0)|_{L^2}^2+(\mathcal{E}_0^u+C_1\mathcal{E}_0^\theta)t
\end{aligned}\tag{2.1}$$

and for $p>1$ and some constant $C_0>0$,

$$\begin{aligned}
&\mathbb{E}|u(t)|_{L^2}^{2p}+C_1\mathbb{E}|\theta(t)|_{L^2}^{2p}+p\nu\int_0^t\mathbb{E}|\Lambda u(s)|_{L^2}^2|u(s)|_{L^2}^{2(p-1)}\,ds\\
&\quad+p\kappa C_1\int_0^t\mathbb{E}|\nabla\theta(s)|_{L^2}^2|\theta(s)|_{L^2}^{2(p-1)}\,ds\\
&\leqslant \mathbb{E}|u(0)|_{L^2}^{2p}+C_1\mathbb{E}|\theta(0)|_{L^2}^{2p}+C_0\int_0^t\mathbb{E}|u(s)|_{L^2}^{2(p-1)}\,ds\\
&\quad+C_0C_1\int_0^t\mathbb{E}|\theta(s)|_{L^2}^{2(p-1)}\,ds.
\end{aligned}\tag{2.2}$$

By applying Gronwall's inequality, we obtain the following estimates on the energy moments:

$$\begin{aligned}
\mathbb{E}|u(t)|_{L^2}^2+C_1\mathbb{E}|\theta(t)|_{L^2}^2\leqslant\ & e^{-\min\{\nu,\kappa\}t}(\mathbb{E}|u(0)|_{L^2}^2+C_1\mathbb{E}|\theta(0)|_{L^2}^2)\\
&+\frac{\mathcal{E}_0^u+C_1\mathcal{E}_0^\theta}{\min\{\nu,\,\kappa\}}(1-e^{-\min\{\nu,\kappa\}t})
\end{aligned}\tag{2.3}$$

and for any $p>1$,

$$\begin{aligned}
&\mathbb{E}|u(t)|_{L^2}^{2p}+C_1\mathbb{E}|\theta(t)|_{L^2}^{2p}\\
&\quad\leqslant e^{-\min\{\nu,\kappa\}t}(\mathbb{E}|u(0)|_{L^2}^{2p}+C_1\mathbb{E}|\theta(0)|_{L^2}^{2p})\\
&\qquad+C_0\int_0^t e^{-\min\{\nu,\kappa\}s}(\mathbb{E}|u(s)|_{L^2}^{2(p-1)}+C_1\mathbb{E}|\theta(s)|_{L^2}^{2(p-1)})ds. \qquad (2.4)
\end{aligned}$$

For the analysis of the enstrophy moments, we need L^q estimates of temperature θ. Applying Ito's formula to the map $\theta\to|\theta|^q$ produces

$$\begin{aligned}
d|\theta|^q =\ &(-u\cdot\nabla|\theta|^q+q\kappa|\theta|^{q-2}\theta\Delta\theta+\frac{q(q-1)}{2}|\theta|^{q-2}(\sum|\tilde{\sigma}_k|^2|e^{ik\cdot x}|^2))\,dt\\
&+q|\theta|^{q-2}\theta dW_\theta.
\end{aligned}$$

Define stopping time $T=\inf\{t\geqslant 0:|\theta(t)|_{L^{2(q-1)}}^{q-1}\geqslant H^{q-1}\}$. By integrating over time interval $[0,\ t\wedge T]$, it follows that:

$$\begin{aligned}
|\theta(t\wedge T)|^q \leqslant\ &|\theta(0)|^q+q\int_0^{t\wedge T}|\theta|^{q-2}\theta\,dW_\theta(s)\\
&+\int_0^{t\wedge T}(-u\cdot\nabla|\theta|^q+\kappa q|\theta|^{q-2}\theta\Delta\theta+\frac{q(q-1)}{2}\\
&\times|\theta|^{q-2}(\sum|\tilde{\sigma}_k|^2))\,ds.
\end{aligned}$$

Integrating over $\mathbb{T}^2$ and using the Fubini Theorem, we obtain that

$$\begin{aligned}
&|\theta(t\wedge T)|_{L^q}^q+\frac{4(q-1)}{q^2}\kappa\int_0^{t\wedge T}|\nabla|\theta(s)|^{(q/2)}|_{L^2}^2\,ds\\
&\leqslant|\theta(0)|_{L^q}^q+\frac{q(q-1)}{2}(\sum|\tilde{\sigma}_k|^2)\int_0^{t\wedge T}|\theta(s)|_{L^{q-2}}^{q-2}\,ds\\
&\quad+q\int_0^t\langle|\theta|^{q-2}\theta,dW_\theta\rangle_{L^2}.
\end{aligned}$$

By taking expectation on the both sides of the above inequality and using optional stopping time lemma, it is immediate that

$$\begin{aligned}
&\mathbb{E}|\theta(t)|_{L^q}^q+\frac{4(q-1)}{q^2}\kappa\int_0^t\mathbb{E}|\nabla|\theta|^{q/2}|_{L^2}^2\,ds\\
&\quad\leqslant\mathbb{E}|\theta(0)|_{L^q}^q+\frac{q(q-1)}{2}(\sum|\tilde{\sigma}_k|^2)\int_0^t\mathbb{E}|\theta(s)|_{L^{q-2}}^{q-2}\,ds. \qquad (2.5)
\end{aligned}$$

Thus we have expectation L^q estimates of the temperature θ.
For the enstrophy estimates we consider the following Ito equations:

$$\begin{aligned}d|\Lambda u(t)|_{L^2}^{2p} = {}& 2p|\Lambda u(t)|_{L^2}^{2(p-1)}[-\nu|\Lambda^2 u(t)|_{L^2}^2\,dt - \sigma\langle\Lambda a(\theta(t)),\,\Lambda u(t)\rangle_{L^2}\,dt\\ &+\langle\Lambda^2 u(t),\,dW_u\rangle_{L^2}] + 2p(p-1)|\Lambda u(t)|_{L^2}^{2(p-2)}\\ &\times\Big(\sum|k|^2|u_k(t)|^2|\sigma_k|^2\Big)dt\\ &+p|\Lambda u(t)|_{L^2}^{2(p-1)}\mathcal{E}_1^{uM}\,dt,\end{aligned}$$

and

$$\begin{aligned}d|\nabla\theta(t)|_{L^2}^{2p} = {}& 2p|\nabla\theta(t)|_{L^2}^{2(p-1)}[-\kappa|\Delta\theta(t)|_{L^2}^2\,dt - \langle u\cdot\nabla\theta,\,\Delta\theta\rangle_{L^2}\,dt\\ &+\langle\Delta\theta(t),dW_\theta\rangle_{L^2}] + 2p(p-1)|\nabla\theta(t)|_{L^2}^{2(p-2)}\\ &\times\Big(\sum_k|k|^2|\theta_k(t)|^2|\tilde{\sigma}_k|^2\Big)\,dt\\ &+p|\nabla\theta(t)|_{L^2}^{2(p-1)}\mathcal{E}_1^{\theta M}\,dt.\end{aligned}$$

Define the stopping time $T=\inf\{t\geqslant 0: |\Lambda^2 u(t)|^2\geqslant H^2 \text{ or } |\Delta\theta(t)|^2\geqslant H^2\}$.
First we consider $p=1$ case.
We begin the enstrophy estimates by providing following interpolation inequality for q, r satisfying $1<r=(2q-4)/(q+2)$:

$$\begin{aligned}C|\nabla\theta|_{L^{2r}}^4 &\leqslant C|\Delta\theta|_{L^2}^{(4r(q+2)-4q)/(r(q+2))}|\theta|_{L^q}^{4q/r(q+2)}\\ &\leqslant \frac{\kappa C_1}{2}|\Delta\theta|_{L^2}^2 + C|\theta|_{L^q}^q.\end{aligned}$$

We also remark that $|\Lambda a(\theta)|_{L^2}^2\leqslant C|\nabla\theta|_{L^2}^2$ for some constant $C>0$. Integrating over the time interval $[0,\,t\wedge T)$ and using Young's inequality and the above inequalities(we take q and r to satisfy above condition), then we have

$$\begin{aligned}&|\Lambda u(t\wedge T)|_{L^2}^2 + 2\nu\int_0^{t\wedge T}|\Lambda^2 u(s)|_{L^2}^2\,ds\\ &\leqslant |\Lambda u(0)|_{L^2}^2 + 2\int_0^{t\wedge T}\langle\Lambda^2 u(s),\,dW_u\rangle_{L^2}\\ &\quad+\nu\int_0^{t\wedge T}|\Lambda u(s)|_{L^2}^2 ds + \frac{\kappa C_1}{2}\int_0^{t\wedge T}|\nabla\theta(s)|_{L^2}^2\,ds + \mathcal{E}_1^{uM}(t\wedge T)\end{aligned}$$

and

$$
\begin{aligned}
&C_1|\nabla\theta(t\wedge T)|_{L^2}^2 + 2\kappa C_1\int_0^{t\wedge T}|\Delta\theta(s)|_{L^2}^2\,ds\\
&\quad\leqslant C_1|\nabla\theta(0)|_{L^2}^2 + C_1\int_0^{t\wedge T}\langle\Delta\theta(s),\,dW_\theta\rangle_{L^2}\\
&\qquad + C_1\int_0^{t\wedge T}|\langle(u\cdot\nabla\theta),\,\Delta\theta\rangle_{L^2}|\,ds + C_1\mathcal{E}_1^{\theta M}(t\wedge T)\\
&\quad\leqslant C_1|\nabla\theta(0)|_{L^2}^2 + C_1\int_0^{t\wedge T}\langle\Delta\theta(s),\,dW_\theta\rangle_{L^2} + \frac{\nu}{2}\int_0^{t\wedge T}|\nabla u|_{L^{\frac{r}{r-1}}}^2\,ds\\
&\qquad + C\int_0^{t\wedge T}|\nabla\theta|_{L^{2r}}^4\,ds + C_1\mathcal{E}_1^{\theta M}(t\wedge T)\\
&\quad\leqslant C_1|\nabla\theta(0)|_{L^2}^2 + C_1\int_0^{t\wedge T}\langle\Delta\theta(s), dW_\theta(s)\rangle_{L^2} + \frac{\nu}{2}\int_0^{t\wedge T}|\Delta u(s)|_{L^2}^2\,ds\\
&\qquad + \frac{\kappa C_1}{2}\int_0^{t\wedge T}|\Delta\theta(s)|_{L^2}^2\,ds + C\int_0^{t\wedge T}|\theta|_{L^q}^q\,ds + C_1(t\wedge T)\mathcal{E}_1^{\theta M}.
\end{aligned}
$$

Taking expectation on the both sides, adding above two inequalities, and using Optional Stopping Lemma($H\to\infty,\ T\to\infty$), we obtain

$$
\begin{aligned}
&\mathbb{E}|\Lambda u(t)|_{L^2}^2 + C_1\mathbb{E}|\nabla\theta(t)|_{L^2}^2 + \nu\int_0^t\mathbb{E}|\Lambda^2u(s)|_{L^2}^2\,ds + \kappa C_1\int_0^t\mathbb{E}|\Delta\theta(s)|_{L^2}^2\,ds\\
&\quad\leqslant\mathbb{E}|\Lambda u(0)|_{L^2}^2 + C_1\mathbb{E}|\nabla\theta(0)|_{L^2}^2 + C\mathbb{E}|\theta(t)|_{L^q}^q t + (\mathcal{E}_1^u + C_1\mathcal{E}_1^\theta)t. \qquad (2.6)
\end{aligned}
$$

For the general p case, we proceed as the following. First we note that for small $\epsilon>0$,(e.g., we take $q=2p+4$)

$$
\begin{aligned}
&|\nabla\theta|_{L^2}^{2(p-1)}(|\nabla\theta|_{L^{4p/(p+1)}}^2|\nabla u|_{L^{2p/(p-1)}})\\
&\quad\leqslant\epsilon|\nabla u|_{L^2}^{2(p-1)}|\Delta u|_{L^2}^2\\
&\qquad + C|\nabla\theta|_{L^2}^{(4p^2-2p+2)/(2p-1)}|\Delta\theta|_{L^2}^{2(p-1)/2p-1}\\
&\quad\leqslant\epsilon|\nabla u|_{L^2}^{2(p-1)}|\Delta u|_{L^2}^2 + \epsilon|\nabla\theta|_{L^2}^{2(p-1)}|\Delta\theta|_{L^2}^2\\
&\qquad + C|\nabla\theta|_{L^2}^{2(p+1)}\\
&\quad\leqslant\epsilon|\nabla u|_{L^2}^{2(p-1)}|\Delta u|_{L^2}^2 + \epsilon|\nabla\theta|_{L^2}^{2(p-1)}|\Delta\theta|_{L^2}^2\\
&\qquad + C|\theta|_{L^q}^{\frac{q}{2}}|\Delta\theta|_{L^2}|\nabla\theta|_{L^2}^{p-1}\\
&\quad\leqslant\epsilon|\nabla u|_{L^2}^{2(p-1)}|\Delta u|_{L^2}^2 + 2\epsilon|\nabla\theta|_{L^2}^{2(p-1)}|\Delta\theta|_{L^2}^2 + C|\theta|_{L^q}^q.
\end{aligned}
$$

Using this kind of interpolation inequality and proceeding similarly to $p=1$ case, we have

$$
\begin{aligned}
&\mathbb{E}|\Lambda u(t)|_{L^2}^{2p} + C_1\mathbb{E}|\nabla\theta(t)|_{L^2}^{2p} \\
&\quad + p\nu\int_0^t \mathbb{E}|\Lambda u(s)|_{L^2}^{2(p-1)}|\Lambda^2 u(s)|_{L^2}^2\,ds + \kappa p C_1 \\
&\quad \times \int_0^t \mathbb{E}|\nabla\theta(s)|_{L^2}^{2(p-1)}|\Delta\theta(s)|_{L^2}^2\,ds \\
&\leqslant \mathbb{E}|\Lambda u(0)|_{L^2}^{2p} + C_1\mathbb{E}|\nabla\theta(0)|_{L^2}^{2p} + C\mathbb{E}|\theta(t)|_{L^q}^q t \\
&\quad + p\mathcal{E}_1^u\int_0^t \mathbb{E}|\Lambda u|_{L^2}^{2(p-1)}\,ds + pC_1\mathcal{E}_1^\theta\int_0^t \mathbb{E}|\nabla\theta|_{L^2}^{2(p-1)}\,ds < \infty. \qquad (2.7)
\end{aligned}
$$

Thus we obtain the energy and enstrophy estimates.

Lemma 2.1. For some positive constants $C_0(\sigma_{\max}, \tilde{\sigma}_{\max}, \mathcal{E}_i^u, \mathcal{E}_i^\theta, p)$, $C_1(\kappa, \nu, \sigma)$, and C, we have energy estimates (2.1) and (2.2). Using Gronwall's inequality, we have (2.3) and (2.4). For the temperature scalar field θ, we have (2.5). Finally, we have enstrophy estimates (2.6) and (2.7).

The following lemma is helpful for constructing nice future pasts.

Lemma 2.2. Fix any $\delta > 1/2$, and $a \in (0, 1)$. Let $(u(t), \theta(t)) = \phi_t^\omega(u_0, \theta_0)$. There exists a $K_1 > 0$ such that whenever $|u_0|_{L^2}^2 + C_1|\theta_0|_{L^2}^2 < C_0$,

$$
\begin{aligned}
\mathbb{P}\Big\{&|u(t)|_{L^2}^2 + C_1|\theta(t)|_{L^2}^2 + \nu\int_0^t |\Lambda u(s)|_{L^2}^2\,ds + \kappa C_1\int_0^t |\nabla\theta(s)|_{L^2}^2\,ds \\
&\leqslant C_0 + (\mathcal{E}_0^u + C_1\mathcal{E}_0^\theta)t + K_1(t+1)^\delta \quad \text{for all} \quad t \geqslant 0\Big\} \geqslant 1 - a.
\end{aligned}
$$

Proof. The energy inequality reads

$$
\begin{aligned}
&|u(t)|_{L^2}^2 + C_1|\theta(t)|_{L^2}^2 + \nu\int_0^t |\Lambda u(s)|_{L^2}^2\,ds + \kappa C_1\int_0^t |\nabla\theta(s)|_{L^2}^2\,ds \\
&\quad \leqslant C_0 + (\mathcal{E}_0^u + C_1\mathcal{E}_0^\theta)t + \int_0^t \langle u(s), dW_u(s)\rangle_{L^2} + C_1\int_0^t \langle\theta(s), dW_\theta(s)\rangle_{L^2}.
\end{aligned}
$$

Since $|u_0|_{L^2}^2 + C_1|\theta_0|_{L^2}^2 < C_0$, all we need to show is that

$$
\mathbb{P}\left\{M_t^u \leqslant \frac{K_1}{2}(t+1)^\delta \text{ and } C_1 M_t^\theta \leqslant \frac{K_1}{2}(t+1)^\delta \text{ for } t \geqslant 0\right\} \geqslant 1 - a
$$

for K_1 large enough, where $M_t^u = \int_0^t \langle u(s),\, dW_u(s)\rangle_{L^2}$ and $M_t^\theta = \int_0^t \langle \theta(s),\, dW_\theta(s)\rangle_{L^2}$. The quadratic variation $[M^u,\, M^u]_t$ can be calculated and one sees that

$$[M^u,\, M^u]_t \leqslant (\sigma_{\max})^2 \int_0^t |u(s)|_{L^2}^2\, ds$$

and

$$[M^\theta,\, M^\theta]_t \leqslant (\tilde{\sigma}_{\max})^2 \int_0^t |\theta(s)|_{L^2}^2\, ds.$$

Hence

$$([M^u,\, M^u]_t)^p \leqslant \sigma_{\max}^{2p} \left(\int_0^t |u(s)|_{L^2}^2 \right)^p \leqslant \sigma_{\max}^{2p} t^{p-1} \int_0^t |u(s)|_{L^2}^{2p}\, ds$$

and

$$([M^\theta,\, M^\theta]_t)^p \leqslant \tilde{\sigma}_{\max}^{2p} \left(\int_0^t |\theta(s)|_{L^2}^2 \right)^p \leqslant \tilde{\sigma}_{\max}^{2p} t^{p-1} \int_0^t |\theta(s)|_{L^2}^{2p}\, ds.$$

From the above energy estimates, we know if $|u(0)|_{L^2}^2 + C_1|\theta(0)|_{L^2}^2 < C_0$, then there exists a constant $C_p(C_0)$ so that $\mathbb{E}|u(t)|_{L^2}^{2p} + C_1\mathbb{E}|\theta(t)|_{L^2}^{2p} \leqslant C_p$ for all $t \geqslant 0$ and $p \geqslant 1$. Now define the events

$$A_k = \left\{ \sup_{s\in[0,\,k]} |M_s^u| > \frac{K_1}{2}(k+1)^\delta \right\}$$

and

$$B_k = \left\{ \sup_{s\in[0,\,k]} C_1|M_s^\theta| > \frac{K_1}{2}(k+1)^\delta \right\}.$$

By the Doob–Kolmogorov martingale inequality, we obtain

$$\mathbb{P}\{A_k\} \leqslant \frac{2^{2p}\mathbb{E}([M^u,\, M^u]_t)^p}{K_1^{2p}(k+1)^{2p\delta}} \leqslant \frac{\sigma_{\max}^{2p} C_p}{K_1^{2p}} \frac{2^{2p}k^p}{(k+1)^{2p\delta}}$$

and

$$\mathbb{P}\{B_k\} \leqslant \frac{(2C_1)^{2p}\mathbb{E}([M^\theta, M^\theta]_t)^p}{K_1^{2p}(k+1)^{2p\delta}} \leqslant \frac{\tilde{\sigma}_{\max}^{2p} C_p}{K_1^{2p}} \frac{(2C_1)^{2p} k^p}{(k+1)^{2p\delta}}.$$

Observe that

$$\mathbb{P}\left\{M_t^u \leqslant \frac{K_1}{2} t^\delta\right\} \geqslant 1 - \mathbb{P}\{\cup_k A_k\} \geqslant 1 - \sum_k \mathbb{P}\{A_k\}$$

and

$$\mathbb{P}\left\{C_1 M_t^\theta \leqslant \frac{K_1}{2} t^\delta\right\} \geqslant 1 - \mathbb{P}\{\cup_k B_k\} \geqslant 1 - \sum_k \mathbb{P}\{B_k\}.$$

By the previous estimate on $\mathbb{P}\{A_k\}$ and $\mathbb{P}\{B_k\}$, for any $\delta > 1/2$ we see that the sum is finite for p sufficiently large. Lastly we note that

$$\mathbb{P}\left\{M_t^u \leqslant \frac{K_1}{2} t^\delta \text{ and } C_1 M_t^\theta \leqslant \frac{K_1}{2} t^\delta \text{ for } t \geqslant 0\right\}$$
$$\geqslant (1 - \sum_k \mathbb{P}\{A_k\})(1 - \sum_k \mathbb{P}\{B_k\}).$$

$\sum \mathbb{P}\{A_k\}$ and $\sum \mathbb{P}\{B_k\}$ can be made arbitrary small by increasing K_1. This completes the proof. ∎

Lemma 2.3. For any stationary measure, all energy moments are finite. In fact for any $p \geqslant 1$, there exists a constant $K_p < \infty$ such that

$$\int_{\mathbb{L}^2} |u|_{L^2}^{2p} + C_1 |\theta|_{L^2}^{2p} \, d\mu(u, \theta) < K_p$$

for all stationary measure μ and the constant C_1 as in the energy estimates. In particular, $K_1 = (\mathcal{E}_0^u + C_1 \mathcal{E}_0^\theta)/(\min\{\nu, \kappa\})$.

Proof. We will consider the case when $p = 1$. The other cases follow by the same method. For any $\epsilon > 0$, there exists a b_ϵ such that

$\mu\{(u,\theta)\in\mathbb{L}^2 : |u|_{L^2}^2 + C_1|\theta|_{L^2}^2 \leqslant b_\epsilon\} > 1-\epsilon$. Let B_ϵ denote $\{(u,\theta)\in\mathbb{L}^2 : |u|_{L^2}^2 + C_1|\theta|_{L^2}^2 \leqslant b_\epsilon\}$. For any $H>0$ and $t>0$, we have

$$\begin{aligned}
&\int_{\mathbb{L}^2}((|u|_{L^2}^2 + C_1|\theta|_{L^2}^2)\wedge H)d\mu(u,\theta)\\
&\quad=\int_{\mathbb{L}^2}\mathbb{E}((|\phi_{0,t}^{\omega}u|_{L^2}^2 + C_1|\phi_{0,t}^{\omega}\theta|_{L^2}^2)\wedge H)d\mu(u,\theta)\\
&\quad\leqslant H\epsilon + \int_{B_\epsilon}\mathbb{E}((|\phi_{0,t}^{\omega}u|_{L^2}^2 + C_1|\phi_{0,t}^{\omega}\theta|_{L^2}^2)\wedge H)d\mu(u,\theta)\\
&\quad\leqslant H\epsilon + \int_{B_\epsilon}\mathbb{E}(|\phi_{0,t}^{\omega}u|_{L^2}^2 + C_1|\phi_{0,t}^{\omega}\theta|_{L^2}^2)d\mu(u,\theta).
\end{aligned}$$

Applying the bound in energy estiamtes gives

$$\begin{aligned}
\int_{\mathbb{L}^2}((|u|_{L^2}^2 + C_1|\theta|_{L^2}^2)\wedge H)d\mu(u,\theta) \leqslant\ & H\epsilon + \frac{\mathcal{E}_0^u + C_1\mathcal{E}_0^\theta}{\min\{\nu,\kappa\}}\\
&+e^{-\min\{\nu,\kappa\}t}\left(b_\epsilon - \frac{\mathcal{E}_0^u + C_1\mathcal{E}_0^\theta}{\min\{\nu,\kappa\}}\right).
\end{aligned}$$

Taking the limit as $t\to\infty$ and then observing that ϵ was arbitrary, we obtain

$$\begin{aligned}
&\int_{\mathbb{L}^2}((|u|_{L^2}^2 + C_1|\theta|_{L^2}^2)\wedge H)d\mu(u,\theta)\\
&\quad\leqslant \frac{\mathcal{E}_0^u + C_1\mathcal{E}_0^\theta}{\min\{\nu,\kappa\}}.
\end{aligned}$$

Taking $H\to\infty$ gives that the energy of any stationary measure is bounded by $(\mathcal{E}_0^u + C_1\mathcal{E}_0^\theta)/\min\{\nu,\kappa\}$. The argument for higher moments of the energy is the same. This completes the proof. ∎

Lemma 2.4. For any stationary measure μ, we have

$$\nu\mathbb{E}\int_{\mathbb{L}^2}|\Lambda u(s)|_{L^2}^2\,ds + \kappa C_1\mathbb{E}\int_{\mathbb{L}^2}|\nabla\theta(s)|_{L^2}^2\,ds \leqslant \mathcal{E}_0^u + C_1\mathcal{E}_0^\theta.$$

Proof. From the energy inequality, we have for any initial condition $(u_0,\theta_0)\in\mathbb{L}^2$,

$$\begin{aligned}
&\mathbb{E}|\phi_{0,t}u_0|^2_{L^2} + C_1\mathbb{E}|\phi_{0,t}\theta_0|^2_{L^2} \\
&\quad + \nu\mathbb{E}\int_0^t |\Lambda\phi_{0,s}u_0|^2_{L^2}\,ds + \kappa C_1\mathbb{E}\int_0^t |\nabla\phi_{0,s}\theta_0|^2_{L^2}\,ds \\
&\leqslant \mathbb{E}|u_0|^2_{L^2} + C_1\mathbb{E}|\theta_0|^2_{L^2} + (\mathcal{E}_0^u + C_1\mathcal{E}_0^\theta)t.
\end{aligned}$$

Here we have switched the time integral and the expectation by the Fubini–Tonelli theorem the integrand is nonnegative. Hence averaging with respect to the stationary measure gives

$$\begin{aligned}
&\int_{\mathbb{L}^2}\mathbb{E}|\phi_{0,t}u_0|^2_{L^2}d\mu(u_0,\theta_0) + C_1\int_{\mathbb{L}^2}\mathbb{E}|\phi_{0,t}\theta_0|^2_{L^2}\,d\mu(u_0,\theta_0) \\
&\quad + \nu\int_{\mathbb{L}^2}\int_0^t \mathbb{E}|\Lambda\phi_{0,s}u_0|^2_{L^2}\,ds\,d\mu(u_0,\theta_0) \\
&\quad + \kappa C_1\int_{\mathbb{L}^2}\int_0^t \mathbb{E}|\nabla\phi_{0,s}\theta_0|^2_{L^2}\,ds\,d\mu(u_0,\theta_0) \\
&\leqslant \mathbb{E}|u(0)|^2_{L^2} + C_1\mathbb{E}|\theta(0)|^2_{L^2} + (\mathcal{E}_0^u + C_1\mathcal{E}_0^\theta)t.
\end{aligned}$$

Because μ was stationary measure, we have

$$\begin{aligned}
&\int_{\mathbb{L}^2}\mathbb{E}|\phi_{0,t}u_0|^2_{L^2}\,d\mu(u_0,\theta_0) + \int_{\mathbb{L}^2} C_1\mathbb{E}|\phi_{0,t}\theta_0|^2_{L^2}\,d\mu(u_0,\theta_0) \\
&\quad = \int_{\mathbb{L}^2}|u_0|^2_{L^2} + C_1|\theta_0|^2_{L^2}d\mu(u_0,\theta_0)
\end{aligned}$$

and

$$\begin{aligned}
&\nu\int_{\mathbb{L}^2}\int_0^t \mathbb{E}|\Lambda\phi_{0,s}u_0|^2_{L^2}\,ds + \kappa C_1\int_{\mathbb{L}^2}\int_0^t \mathbb{E}|\nabla\phi_{0,s}\theta_0|^2_{L^2}\,ds \\
&\quad = \nu t\int_{\mathbb{L}^2}|\Lambda u_0|^2_{L^2}\,d\mu(u_0,\theta_0) + \kappa C_1 t\int_{\mathbb{L}^2}|\nabla\theta_0|^2_{L^2}\,d\mu(u_0,\theta_0).
\end{aligned}$$

This completes the proof. ∎

Lemma 2.5. Let μ_p be the measure induced on $C((-\infty,0],\mathbb{L}^2)$ by any given stationary measure μ. Fix any $K_0 > 0$ and $\delta > 1/2$. Then for μ_p-almost every trajectory $(u(s),\theta(s))$ in $C((-\infty,0],\mathbb{L}^2)$, there exists a constant T such that for $s \leqslant 0$, $|u(s)|^2_{L^2} + C_1|\theta(s)|^2_{L^2} \leqslant \mathcal{E}_0^u + C_1\mathcal{E}_0^\theta + K_0\min(T,|s|)^\delta$.

Proof. The basic energy estimate reads

$$
\begin{aligned}
&|u(t)|_{L^2}^2 + C_1|\theta(t)|_{L^2}^2 + \nu\int_{t_0}^t |\Lambda u(s)|_{L^2}^2\,ds + \kappa C_1\int_{t_0}^t |\nabla\theta(s)|_{L^2}^2\,ds\\
&\quad\leqslant |u(t_0)|_{L^2}^2 + C_1|\theta(t_0)|_{L^2}^2 + (\mathcal{E}_0^u + C_1\mathcal{E}_0^\theta)(t-t_0)\\
&\qquad + \int_{t_0}^t \langle u(s),\, dW_u(s)\rangle_{L^2} + C_1\int_{t_0}^t \langle\theta(s),\, dW_\theta(s)\rangle_{L^2}.
\end{aligned}
$$

For any $t_0 < t \leqslant 0$, it can be shown that there is no problem writing the integration against the Wiener path in the above integral (for details see ref. 6). For any $k \geqslant 1$, we have the following from the above inequality

$$
\begin{aligned}
&\sup_{s\in[-k,\,-k+1]} |u(s)|_{L^2}^2 + C_1 \sup_{s\in[-k,\,-k+1]} |\theta(s)|_{L^2}^2\\
&\quad\leqslant |u(-k)|_{L^2}^2 + C_1|\theta(-k)|_{L^2}^2 + \mathcal{E}_0^u + C_1\mathcal{E}_0^\theta + \sup_{s\in[-k,\,-k+1]} F_k(s),
\end{aligned}
$$

where

$$
\begin{aligned}
F_k(s) = &-\nu\int_{-k}^s |\Lambda u(r)|_{L^2}^2\,dr - \kappa C_1\int_{-k}^s |\nabla\theta(r)|_{L^2}^2\,dr + \int_{-k}^s \langle u(r),\, dW_u(r)\rangle_{L^2}\\
&+ C_1\int_{-k}^s \langle\theta(r),\, dW_\theta(r)\rangle_{L^2}.
\end{aligned}
$$

Now define

$$
\begin{aligned}
A_k = \{(u(s),\,\theta(s)) :\ &\sup_{s\in[-k,\,-k+1]} |u(s)|_{L^2}^2 + \sup_{s\in[-k,\,-k+1]} C_1|\theta(s)|_{L^2}^2\\
&\leqslant \mathcal{E}_0^u + C_1\mathcal{E}_0^\theta + K_0|k-1|^\delta\}
\end{aligned}
$$

and $U_T = \cap_{k>T} A_k$. Since U_T are an increasing collection of sets, it will be sufficient to prove that $\lim_{T\to\infty}\mu_p(U_T) = 1$. Now since $\mu_p(U_T^c) \leqslant \sum_{k>T}\mu_p(A_k^c)$, we need only to show that $\sum_{k>0}\mu_p(A_k^c) < \infty$. We have

$$
\begin{aligned}
\mu_p(A_k^c) \leqslant\ &\mu_p\{(u(s),\,\theta(s)) : |u(-k)|_{L^2}^2 + C_1|\theta(-k)|_{L^2}^2 \geqslant \frac{K_0}{2}|k-1|^\delta\}\\
&+ \mu_p\{(u(s),\,\theta(s)) : \sup_{s\in[-k,\,-k+1]} F_k(s) \geqslant \frac{K_0}{2}|k-1|^\delta\}.
\end{aligned}
$$

For the first term of the above inequality, Chebyshev's inequality and Lemma 2.3 produce

$$
\begin{aligned}
&\mu_p\{(u(s),\theta(s)) : |u(-k)|_{L^2}^2 + C_1|\theta(-k)|_{L^2}^2 \geqslant \frac{K_0}{2}|k-1|^\delta\} \\
&\leqslant \frac{16}{K_0^2|k-1|^{2\delta}}(\mathbb{E}|u(-k)|_{L^2}^4 + C_1^2\mathbb{E}|\theta(-k)|_{L^2}^4) \\
&\leqslant \frac{16C}{K_0^2|k-1|^{2\delta}},
\end{aligned}
$$

which is summable if $\delta > 1/2$.

For the second term, we control $F_k(s)$ by estimating $M_k^u(s) - \alpha[M_k^u,\, M_k^u](s)$ and $M_k^\theta(s) - \beta[M_k^\theta,\, M_k^\theta](s)$. We have

$$
\begin{aligned}
[M_k^u,\, M_k^u](s) &= \int_{-k}^s \sum_l |\sigma_l|^2 |u_l(r)|^2\, dr \leqslant (\sigma_{\max})^2 \int_{-k}^s |u(r)|_{L^2}^2\, dr \\
&\leqslant (\sigma_{\max})^2 \int_{-k}^s |\Lambda u(r)|_{L^2}^2\, dr
\end{aligned}
$$

and

$$
\begin{aligned}
[M_k^\theta,\, M_k^\theta](s) &= \int_{-k}^s \sum_l |\tilde{\sigma}_l|^2 |\theta_l(r)|_{L^2}^2\, dr \leqslant (\tilde{\sigma}_{\max})^2 \int_{-k}^s |\theta(r)|_{L^2}^2\, dr \\
&\leqslant (\tilde{\sigma}_{\max})^2 \int_{-k}^s |\nabla\theta(r)|_{L^2}^2\, dr.
\end{aligned}
$$

Hence $F_k(s) \leqslant M_k^u(s) + C_1 M_k^\theta(s) - (2\nu/(\sigma_{\max})^2)[M_k^u,\, M_k^u] - (2\kappa C_1/(\tilde{\sigma}_{\max})^2)$ $[M_k^\theta,\, M_k^\theta](s)$ almost surely. The exponential martingale inequality gives us that

$$
\begin{aligned}
&\mu_p\left\{(u(s),\, \theta(s)) : \sup_{s\in[-k,\, -k+1]} F_k(s) \geqslant \frac{K_0}{2}|k-1|^\delta\right\} \\
&\leqslant \exp\left(-\frac{2\nu K_0}{(\sigma_{\max})^2}|k-1|^\delta\right) + \exp\left(-\frac{2\kappa K_0}{(\tilde{\sigma}_{\max})^2 C_1}|k-1|^\delta\right).
\end{aligned}
$$

Since this is summable for any $\delta > 0$, the proof is completed. ∎

3. PROOF OF THEOREM 1

Define two subspaces of L^2

$$L_\ell^2 = \mathrm{span}\{e_k,\ |k| \leqslant N\},\ L_h^2 = \mathrm{span}\{e_k,\ |k| > N\}$$

and two subspaces of $\tilde{L}^2$

$$\tilde{L}_\ell^2 = \mathrm{span}\{\tilde{e}_k,\ |k| \leqslant N\},\ \tilde{L}_h^2 = \mathrm{span}\{\tilde{e}_k,\ |k| > N\}.$$

We will call L_ℓ^2 the set of low modes and L_h^2 the set of high modes. We denote $\mathbb{L}_\ell^2 = L_\ell^2 \oplus \tilde{L}_\ell^2$ and $\mathbb{L}_h^2 = L_h^2 \oplus \tilde{L}_h^2$. We sometimes denote $\tilde{L}_\ell^2$ and $\tilde{L}_h^2$ by L_ℓ^2 and L_h^2 for simplicity. Obviously, $L^2 = L_\ell^2 \oplus L_h^2$. Denote by P_ℓ and P_h the projections onto the low and high mode spaces. Since we are concerned with stationary measure, we are interested in stationary solutions of (1.2) that exists for time from $-\infty$ to ∞. We will show in this section that for such solutions, the high modes are completely determined by the past history of the low modes. For this purpose, we have

$$\begin{aligned} d\ell^u(t) &= [-\nu\Lambda^2 \ell^u + P_\ell B(\ell^u,\ \ell^u)]\,dt \\ &\quad + [P_\ell B(\ell^u, h^u) + P_\ell B(h^u, \ell^u) + P_\ell B(h^u, h^u)]dt + \sigma a(\ell^\theta) + dW_u(t), \end{aligned} \tag{3.1}$$

$$\begin{aligned} d\ell^\theta(t) &= [\kappa\Delta\ell^\theta + P_\ell(\ell^u \cdot \nabla \ell^\theta)]dt \\ &\quad + [P_\ell(\ell^u \cdot \nabla h^\theta) + P_\ell(h^u \cdot \nabla \ell^\theta) + P_\ell(h^u \cdot \nabla h^\theta)]dt + dW_\theta(t), \end{aligned} \tag{3.2}$$

$$\begin{aligned} \frac{dh^u(t)}{dt} &= [-\nu\Lambda^2 h^u + P_h B(h^u,\ h^u)] \\ &\quad + [P_h B(\ell^u,\ h^u) + P_h B(h^u,\ \ell^u) + P_h B(h^u,\ h^u)] + \sigma a(h^\theta) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \frac{dh^\theta(t)}{dt} &= [\kappa\Delta h^\theta + P_h(h^u \cdot \nabla h^\theta)] \\ &\quad + [P_h(\ell^u \cdot \nabla h^\theta) + P_h(h^u \cdot \nabla \ell^\theta) + P_h(h^u \cdot \nabla h^\theta)]. \end{aligned} \tag{3.4}$$

Define the set of nice pasts $U \subset C((-\infty,\ 0], \mathbb{L}^2)$ to consist of all $(u,\ \theta) : (-\infty,\ 0] \to \mathbb{L}^2$ such that

(i) $u(t)$ and $\theta(t)$ are in H^1 for all $t \leqslant 0$,

(ii) the energy averages correctly. More precisely,

$$\lim_{t\to-\infty}\frac{\nu}{|t|}\int_t^0 |\Lambda u(s)|_{L^2}^2\,ds+\lim_{t\to-\infty}\frac{\kappa C_1}{|t|}\int_t^0 |\nabla\theta(s)|_{L^2}^2\,ds\leqslant \mathcal{E}_0^u+C_1\mathcal{E}_0^\theta.$$

(iii) the energy fluctuations are typical. More precisely, there exists a $T=T(u)$ such that

$$|u(t)|_{L^2}^2+C_1|\theta(t)|_{L^2}^2\leqslant \mathcal{E}_0^u+C_1\mathcal{E}_0^\theta+\max(|t|,\,T)^{2/3}.$$

Lemma 3.1. Let μ_p be the measure on $C((-\infty,0],\mathbb{L}^2)$ induced by a stationary measure μ for (1.2). Then $\mu_p(U)=1$.

Proof. In Section 2, we prove solutions of the Boussinesq equations are in H^1 for all t. The fact that the last condition is satisfied by a set of full measure is proved in Lemma 2.5. All that remains to show is (ii). From Lemma 2.4, we have $|\Lambda u|_{L^2}^2,\ |\nabla\theta|_{L^2}^2\in L^1(\mu)$ for any stationary measure μ and $\nu\int|\Lambda u|_{L^2}^2\,d\mu+\kappa C_1\int|\nabla\theta|_{L^2}^2\,d\mu\leqslant\mathcal{E}_0^u+C_1\mathcal{E}_0^\theta$. Since the measure is invariant under shifts back in time, the ergodic theorem implies that for μ_p-almost every trajectories time average converges to the average of $|\Lambda u|_{L^2}^2$ and $|\nabla\theta|_{L^2}^2$. ∎

By $\Phi_{s,t}(\ell,\,h_0)$, we mean the solution to (3.3) and (3.4) at time t given the initial condition h_0 at time s and the "forcing" $\ell=(\ell^u,\,\ell^\theta)$. Denote by $\mathcal{P}$ the set of all $\ell\in C((-\infty,0],\mathbb{L}_\ell^2)$ such that $\ell=P_\ell(u,\theta)$ and $h(t)=\Phi_{s,t}(\ell,\,h(s))$ for any $s<t\leqslant 0$. From Lemma 3.1, $\mathcal{P}$ is not empty.

Lemma 3.2. Suppose there exists a positive sufficiently large constant $\hat{C}=\hat{C}(\nu,\kappa)$ and N is so large that $N^2>\hat{C}(\mathcal{E}_0^u+C_1\mathcal{E}_0^\theta)$. If there exists two solutions

$$\begin{pmatrix}u_1\\ \theta_1\end{pmatrix}=\begin{pmatrix}\ell^u(t)+h_1^u(t)\\ \ell^\theta(t)+h_1^\theta(t)\end{pmatrix},\ \begin{pmatrix}u_2\\ \theta_2\end{pmatrix}=\begin{pmatrix}\ell^u(t)+h_2^u(t)\\ \ell^\theta(t)+h_2^\theta(t)\end{pmatrix}$$

corresponding to some realizations of the forcing and such that

$$\begin{pmatrix}u_1\\ \theta_1\end{pmatrix},\ \begin{pmatrix}u_2\\ \theta_2\end{pmatrix}\in U,$$

then $u_1=u_2$, $\theta_1=\theta_2$, i.e., $h_1^u=h_2^u$ and $h_1^\theta=h_2^\theta$. Furthermore, given a solution $(u(t),\,\theta(t))\in U$, any h_0, and $t\leqslant 0$ the following limit exists:

$$\lim_{t_0\to-\infty}\Phi_{t_0,t}(\ell,\,h_0)=h^*$$

and $h^*=h(t)$.

Proof. Denote by $\rho^u(t) = h_1^u(t) - h_2^u(t)$ and $\rho^\theta(t) = h_1^\theta(t) - h_2^\theta(t)$. We have

$$\frac{d\rho^u}{dt} = -\nu\Lambda^2\rho^u + P_h B(u_1, \rho^u) + P_h B(\rho^u, u_2) + \sigma a(\rho^\theta)$$

and

$$\frac{d\rho^\theta}{dt} = \kappa\Delta\rho^\theta + P_h(u_1 \cdot \nabla\rho^\theta) + P_h(\rho^u \cdot \nabla\theta_2).$$

Taking inner product with ρ^u and ρ^θ, respectively, and using the fact that $\langle P_h B(u_1, \rho^u), \rho^u\rangle_{L^2} = \langle P_h B(u_1, \rho^\theta), \rho^\theta\rangle_{L^2} = 0$, and $|a(\rho^\theta)|_{L^2} \leqslant C|\rho^\theta|_{L^2}$ gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(|\rho^u|_{L^2}^2 + \frac{C\sigma^2}{\kappa\nu}|\rho^\theta|_{L^2}^2) \\
&\quad \leqslant -\nu|\Lambda\rho^u|_{L^2}^2 - \frac{C\sigma^2}{\nu}|\nabla\rho^\theta|_{L^2}^2 + \langle P_h B(\rho^u, u_2), \rho^u\rangle_{L^2} \\
&\qquad + \frac{C\sigma^2}{\kappa\nu}\langle P_h(\rho^u \cdot \nabla\theta_2), \rho^\theta\rangle_{L^2} + \sigma\langle a(\rho^\theta), \rho^u\rangle_{L^2} \\
&\quad \leqslant -\frac{\nu}{4}|\Lambda\rho^u|_{L^2}^2 - \frac{C\sigma^2}{4\nu}|\nabla\rho^\theta|_{L^2}^2 + \frac{C}{2\nu}|\rho^u|_{L^2}^2|\Lambda u_2|_{L^2}^2 \\
&\qquad + \frac{C\sigma^4}{\kappa^2\nu^4}|\rho^\theta|_{L^2}^2|\nabla\theta_2|_{L^2}^2 + \frac{C\sigma^2}{\kappa\nu}|\rho^u|_{L^2}^2|\nabla\theta_2|_{L^2}^2.
\end{aligned}$$

Since ρ contains only the modes with $|k| > N$, the Poincare inequality implies that

$$\begin{aligned}
&\frac{d}{dt}\left(|\rho^u|_{L^2}^2 + \frac{C\sigma^2}{\kappa\nu}|\rho^\theta|_{L^2}^2\right) \\
&\quad \leqslant \left(-\min\left(\frac{\nu}{4}, \frac{\kappa}{4}\right)N^2 + \frac{C}{\nu}|\Lambda u_2|_{L^2}^2 + C\frac{\sigma^2(1+\nu^2)}{\kappa\nu^3}|\nabla\theta_2|_{L^2}^2\right) \\
&\qquad \times \left(|\rho^u|_{L^2}^2 + \frac{C\sigma^2}{\kappa\nu}|\rho^\theta|_{L^2}^2\right).
\end{aligned}$$

For $t_0 < t < 0$, we have

$$\begin{aligned}
&\left(|\rho^u(t)|^2_{L^2} + \frac{C\sigma^2}{\kappa\nu}|\rho^\theta(t)|^2_{L^2}\right)\\
&\quad\leqslant \left(|\rho^u(t_0)|^2_{L^2} + \frac{C\sigma^2}{\kappa\nu}|\rho^\theta(t_0)|^2_{L^2}\right)\\
&\quad\quad\times \exp\left\{-\min\left(\frac{\nu}{4}, \frac{\kappa}{4}\right)N^2(t-t_0) + \frac{C}{\nu}\int_{t_0}^t |\Lambda u_2(s)|^2_{L^2}\,ds\right.\\
&\quad\quad\left.+C\frac{\sigma^2(1+\nu^2)}{\kappa\nu^3}\int_{t_0}^t |\nabla\theta_2(s)|^2_{L^2}\,ds\right\}.
\end{aligned}$$

Note that we can choose $\hat{C}$ so large that the exponent is negative. For concreteness, we can choose $\hat{C}$ satisfying

$$\hat{C} > \frac{4\left(\frac{1}{\nu} + \frac{1}{\kappa C_1}\right)\left(\frac{C}{\nu} + \frac{C\sigma^2(1+\nu^2)}{\kappa\nu^3}\right)}{\min\left(\frac{\nu}{4}, \frac{\kappa}{4}\right)}.$$

From the property (ii) of paths in U, we obtain for $t < T_1$ and for some positive constant γ,

$$\begin{aligned}
&-\min\left(\frac{\nu}{4}, \frac{\kappa}{4}\right)N^2(t-t_0) + \frac{C}{\nu}\int_{t_0}^t |\Lambda u_2(s)|^2_{L^2}\,ds\\
&+C\frac{\sigma^2(1+\nu^2)}{\kappa\nu^3}\int_{t_0}^t |\Lambda\theta_2(s)|^2_{L^2}\,ds \leqslant -\frac{\gamma}{2}(t-t_0).
\end{aligned}$$

From the property (iii) of the paths in U, we have $|\rho^u(t)|_{L^2}, |\rho^\theta(t)|_{L^2} \to 0$ as $t_0 \to -\infty$. For the second part of Lemma 3.2, proceed as before letting the given solution $u(t)$ and $\theta(t)$ play the same role of $u_2(t)$ and $\theta_2(t)$ and the solution to (3.3) and (3.4) starting from h_0^u and h_0^θ play the role of u_1 and θ_1, respectively. Hence we have

$$
\begin{aligned}
&\left(|\rho^u(t)|^2_{L^2} + \frac{C\sigma^2}{\kappa\nu}|\rho^\theta(t)|^2_{L^2}\right) \\
&\quad \leqslant \left(|h^u(t_0) - h^u_0|^2_{L^2} + \frac{C\sigma^2}{\kappa\nu}|h^\theta(t_0) - h^\theta_0|^2_{L^2}\right) \\
&\quad \times \exp\left\{-\min\left(\frac{\nu}{4}, \frac{\kappa}{4}\right) N^2(t-t_0)\right. \\
&\quad \left. + \frac{C}{\nu}\int_{t_0}^t |\Lambda u(s)|^2_{L^2} ds + C\frac{\sigma^2(1+\nu^2)}{\kappa\nu^3}\int_{t_0}^t |\nabla\theta(s)|^2_{L^2}\, ds\right\}.
\end{aligned}
$$

The same reasoning as before produces $\rho^u(t)$ and $\rho^\theta(t)$ go to zero as $t_0 \to -\infty$. Therefore, it completes the proof of lemma. ∎

We reduce the dynamics of the Navier–Stokes equations to the dynamics of a finite-dimensional set of low modes with memory. The reduced dynamics is no longer Markovian but rather Gibbsian. Let A be a cylinder set of the type: For some $t_0, t_1, \ldots, t_n$, $t_0 < t_1 < \cdots < t_n \leqslant 0$.

$$
A = \left\{ \begin{pmatrix} u(s) \\ \theta(s) \end{pmatrix} \in C((-\infty, 0], \mathbb{L}^2), \begin{pmatrix} u(t_i) \\ \theta(t_i) \end{pmatrix} \in A_i,\ i = 0, \ldots, n \right\},
$$

where A_i's are Borel sets of L^2.

$$
B = \left\{ (y, \omega),\ y = \begin{pmatrix} u \\ \theta \end{pmatrix} \in A_0,\ \phi^\omega_{t_0, t_1} y \in A_i,\ i = 0, \ldots, n \right\}.
$$

We define $\mu_p(A) = (\mathbb{P} \times \mu)(B)$ where $\mathbb{P} \times \mu$ is the product measure on $\Omega \times \mathbb{L}^2$ and

$$
\psi^\omega_t \begin{pmatrix} u \\ \theta \end{pmatrix}(s) = \phi^\omega_s \begin{pmatrix} u \\ \theta \end{pmatrix}(0) \text{ for } s \in [0, t]
$$

and

$$
\psi^\omega_t \begin{pmatrix} u \\ \theta \end{pmatrix}(s) = \begin{pmatrix} u \\ \theta \end{pmatrix}(s) \text{ for } s \leqslant 0.
$$

Because of Lemma 3.2, we can define a map $\Phi_0 = (\Phi^u_0, \Phi^\theta_0)$, which reconstructs the high modes of the solution at time zero from given low mode trajectories stretching from zero back to $-\infty$. In this notation $h^u(0) =$

$\Phi_0^u(L_u^0)$ and $h^\theta(0) = \Phi_0^\theta(L_\theta^0)$ where L_u^0 and L_θ^0 are some low mode past in $\mathcal{P}$. Define

$$\Phi_t(L^t) = \Phi_t \begin{pmatrix} L_u^t \\ L_\theta^t \end{pmatrix} = \begin{pmatrix} \Phi_t^u(L_u^t, \Phi_0^u(L_u^0)) \\ \Phi_t^\theta(L_\theta^t, \Phi_0^\theta(L_\theta^0)) \end{pmatrix},$$
$$\ell = (\ell^u, \ell^\theta), \text{ and } h = (h^u, h^\theta).$$

Now given any initial low mode past of $L^0 \in \mathcal{P}$, we can solve the future of ℓ^u using the Gibbsian dynamics

$$\begin{aligned} d\ell^u(t) &= [-\nu\Lambda^2\ell^u + P_\ell B(\ell^u, \ell^u) + \sigma a(\ell^\theta) + G_1(\ell^u(t), \Phi_t^u(L_u^t))]dt + dW_u(t), \\ d\ell^\theta(t) &= [\kappa\Delta\ell^\theta + P_\ell(\ell^u \cdot \nabla\ell^\theta) + G_2(\ell, \Phi_t(L^t))]dt + dW_\theta(t), \end{aligned} \tag{3.5}$$

where $G_1(\ell^u, h^u) = P_\ell B(\ell^u, h^u) + P_\ell B(h^u, \ell^u) + P_\ell B(h^u, h^u)$ and $G_2(\ell, h) = P_\ell(\ell^u \cdot \nabla h^\theta) + P_\ell(h^u \cdot \nabla \ell^\theta) + P_\ell(h^u \cdot \nabla h^\theta)$. We also let $Q_t(L^0, \cdot)$ be the measure induced on $C([0, t], \mathbb{L}_\ell^2)$ by the dynamics starting from L^0. Let μ be an ergodic stationary measure and define $L_i^s = S_i^\omega L_i^0$ and $\ell_i(s) = L_i^t(s)$ for $s \leqslant t$. Set $h_i(s) = \Phi_s(L_i^s)$ and $u_i(s) = (\ell_i(s), h_i(s))$. We define the sets

$$\begin{aligned} A_i(K) = \{\vec{f} \in C([0, \infty), \mathbb{L}_\ell^2) :& \\ & |u(t)|_{L^2}^2 + C_1|\theta(t)|_{L^2}^2 + \nu\int_0^t |\Lambda u(s)|_{L^2}^2\,ds + \kappa C_1\int_0^t |\nabla\theta(s)|_{L^2}^2\,ds \\ & < C_0 + (\mathcal{E}_0^u + C_1\mathcal{E}_0^\theta)t + Kt^{4/5}, \\ & \text{where } (u(s), \theta(s)) = \vec{f}(s) + \Phi_s(\vec{f}, h_i)\}. \end{aligned}$$

We set $A(K) = A_1(K) \cap A_2(K)$. By Lemma 2.2, we know that for any $a \in (0, 1)$ there exists a K such that

$$\mathbb{P}\{\omega : S_t^\omega L_i^0 \in A_i(K)\} > 1 - \frac{a}{2} \qquad \text{for } i = 1, 2$$

and hence $\mathbb{P}\{\omega : S_t^\omega L_i^0 \in A(K) \text{ for } i = 1, 2\} > 1 - a > 0$. This is just another way of saying $Q_\infty(L_i^0, A(K)) > 1 - a$.

Lemma 3.3. Let L_1^0 and L_2^0 be two initial pasts in $\mathcal{P}$ such that $L_1^0(0) = L_2^0(0)$. Let $A(K) \subset C([0, \infty), \mathbb{L}_\ell^2)$ be as defined above. For any choice of $K > 0$, $Q_\infty(L_1^0, \cdot \cap A(K))$ is equivalent to $Q_\infty(L_2^0, \cdot \cap A(K))$.

Proof. We consider the following truncated process y, z which will agree with ℓ on the set $A = A(K)$:

$$\begin{aligned}
dy_i(t) = &\ [-\nu\Lambda^2 y_i(t) + P_\ell B(y_i(t),\ y_i(t)) + \sigma a(z_i) \\
&+ \Theta_t(Y_i^t,\ Z_i^t) G_1(y_i(t),\ \Phi_t^u(Y_i^t, h_i^u(0)))]dt + dW_u(t), \\
dz_i(t) = &\ [\kappa \Delta z_i(t) + P_\ell(y_i \cdot \nabla z_i) \\
&+ \Theta_t(Y_i^t,\ Z_i^t) G_2(y(t),\ \Phi_t(Y_i^t, Z_i^t, h_i(0)))]dt + dW_\theta(t), \\
y_i(0) = &\ \ell_i^u(0),\ z_i(0) = \ell_i^\theta(0),
\end{aligned}$$

where $(h_i^u(0),\ h_i^\theta(0)) = \Phi_t(L_i^0)$,

$$\Theta_t(f, g) = \begin{cases} 1 & \text{if } (f, g) \in A|_{[0,\,t]}, \\ 0 & \text{if } (f, g) \notin A|_{[0,\,t]} \end{cases}$$

and $A|_{[0,\,T]}$ is the low mode paths which agree with a path in A up to time T. Recall that $\Phi_t((Y_i^t,\ Z_i^t),\ h_i(0))$ is the solution to with $\ell = (Y,\ Z)$ and $h(0) = h_i(0)$. Let $Q_\infty^{y,z}(L_1^0,\ \cdot)$ and $Q_\infty^{y,z}(L_2^0,\ \cdot)$ be the measures induced by $(Y_1,\ Z_1)$ and $(Y_2,\ Z_2)$, respectively. If applicable, Girsanov's theorem would imply that these measure are equivalent, that is $Q_\infty^{y,z}(L_1^0,\ \cdot)$ and $Q_\infty^{y,z}(L_2^0,\ \cdot)$ be the measures induced by $(Y_1,\ Z_1)$ and $(Y_2,\ Z_2)$, respectively. If applicable, Girsanov's theorem would imply that these measures are equivalent, that is, $Q_\infty^{y,z}(L_1^0,\ \cdot) \equiv Q_\infty^{y,z}(L_2^0,\ \cdot)$. For Girsanov's theorem to apply

$$\mathbb{E}\exp\left\{\int_0^\infty |\Sigma^{-1}\Theta_t(Y_1^t,\ Z_1^t) D(\vec{a},\ \vec{b},\ \vec{c})|^2\, dt\right\} < \infty,$$

where

$$\vec{a} = \begin{pmatrix} y_1(t) \\ z_1(t) \end{pmatrix}, \vec{b} = \begin{pmatrix} \Phi_t^u(Y_1^t,\ h_1^u(0)) \\ \Phi_t^\theta(Z_1^t,\ h_1^\theta(0)) \end{pmatrix}, \vec{c} = \begin{pmatrix} \Phi_t^u(Y_2^t,\ h_2^u(0)) \\ \Phi_t^\theta(Z_2^t,\ h_2^\theta(0)) \end{pmatrix}$$

and

$$\begin{aligned}
&D\left(\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}\right) \\
&\quad = \begin{pmatrix} G_1(h_1,\ f_1) - G_1(h_1,\ f_2) \\ G_2((h_1, h_2),\ (f_1, g_1)) - G_2((h_1, h_2),\ (f_1, g_1)) \end{pmatrix}.
\end{aligned}$$

Also Σ is a 3×3 diagonal matrix with σ_k's and $\tilde{\sigma}_k$'s on its diagonal. Since $|\Sigma^{-1}| < \infty$, it would be enough to show that

$$\sup_{\omega} \int_0^{\infty} |\Theta_t(Y_1^t) D(\vec{a}, \vec{b}, \vec{c})|^2 \, dt < \infty,$$

where $\vec{a}$, $\vec{b}$, $\vec{c}$ are defined as above and $\ell_i(s) = (\ell_i^u(s), \ell_i^\theta(s))$, $h_i(s) = (h_i^u(s), h_i^\theta(s))$. Putting $h_i(s) = \Phi_s((Y_i^s, Z_i^s), h_i(0))$, $(u_i(s), \theta_i(s)) = \ell_i(s) + h_i(s)$, $\rho(s) = h_1(s) - h_2(s)$ and using estimates for the nonlinear terms (see ref. 5), we obtain

$$\begin{aligned} |D(\ell_1(s), h_1(s), h_2(s))|_{L^2}^2 \leqslant{}& C|\rho^u(s)|_{L^2}^2 [|u_1(s)|_{L^2}^2 + |u_2(s)|_{L^2}^2] \\ &+ C|\rho^u(s)|_{L^2}^2 [|\theta_1(s)|_{L^2}^2 + |\theta_2|_{L^2}^2] \\ &+ C|\rho^\theta(s)|_{L^2}^2 [|u_1(s)|_{L^2}^2 + |u_2(s)|_{L^2}^2]. \end{aligned}$$

Notice that if $\ell_i \in A|_{[0,T]}$ then for all $t \in [0, T]$ and some positive constant C, we have

$$|u_i(s)|_{L^2}^2, \; |\theta_i(s)|_{L^2}^2 \leqslant C(C_0 + (\mathcal{E}_0^u + C_1 \mathcal{E}_0^\theta) t + K t^{4/5}),$$

$$\int_0^t |\Lambda u_i(s)|_{L^2}^2 \, ds, \; \int_0^t |\nabla \theta_i(s)|_{L^2}^2 \, ds \leqslant C(C_0 + (\mathcal{E}_0^u + C_1 \mathcal{E}_0^\theta) t + K t^{4/5}).$$

As in the proof of Lemma 3.2, using above estimates produces

$$\begin{aligned} &\left(|\rho^u(t)|_{L^2}^2 + \frac{C\sigma^2}{\kappa\nu} |\rho^\theta(t)|_{L^2}^2 \right) \\ &\quad \leqslant \left(|\rho^u(0)|_{L^2}^2 + \frac{C\sigma^2}{\kappa\nu} |\rho^\theta(0)|_{L^2}^2 \right) \\ &\quad\quad \times \exp\left\{ -\min\left(\frac{\nu}{4}, \frac{\kappa}{4}\right) N^2 t + \frac{C}{\nu} \int_0^t |\Lambda u_2(s)|_{L^2}^2 ds \right. \\ &\quad\quad \left. + C \frac{\sigma^2(1+\nu^2)}{\kappa\nu^3} \int_0^t |\nabla\theta_2(s)|_{L^2}^2 ds \right\} \\ &\quad \leqslant C_0 \exp\left\{ -\min\left(\frac{\nu}{4}, \frac{\kappa}{4}\right) N^2 t + C\left(\frac{1}{\nu} + \frac{\sigma^2(1+\nu^2)}{\kappa\nu^3}\right) \right. \\ &\quad\quad \left. \times \, (C_0 + (\mathcal{E}_0^u + C_1\mathcal{E}_0^\theta) t + K t^{\frac{4}{5}}) \right\}. \end{aligned}$$

Since by assumption $N^2 > \hat{C}(\mathcal{E}_0^u + C_1 \mathcal{E}_0^\theta)$(we can choose $\hat{C}$ to be sufficiently large), hence the estimate on the right hand side of decays exponentially fast. This completes the proof. ▮

For the next step, we need to control high modes by the low modes as the following.

Lemma 3.4. If $h^u(t)$ and $h^\theta(t)$ are the solutions to (3.3), (3.4) with some low mode forcing ℓ^u, $\ell^\theta \in C([0, t], L^2_\ell)$, then $\sup_{s\in[0,t]} |h^u(s)|_{L^2} + \sup_{s\in[0,t]} |h^\theta(s)|_{L^2}$ is bounded by a constant depending on $|h^u(0)|_{L^2}$, $|h^\theta(0)|_{L^2}$, $\int_0^t |\ell^u|^4_{L^2}\, ds$ and $\int_0^t |\ell^\theta|^4_{L^2}\, ds$.

Proof. Taking the inner product with h^u and h^θ, respectively, produces

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|h^u(t)|^2_{L^2} = {} & -\nu|\Lambda h^u(t)|^2_{L^2} + \langle P_h B(h^u, \ell^u), h^u\rangle_{L^2} \\ & + \langle P_h B(\ell^u, \ell^u), h^u\rangle_{L^2} + \langle \sigma a(h^\theta), h^u\rangle_{L^2}\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|h^\theta(t)|^2_{L^2} = {} & -\kappa|\nabla h^\theta|_{L^2} + \langle P_h(h^u\cdot\nabla\ell^\theta), h^\theta\rangle_{L^2} \\ & + \langle P_h(\ell^u\cdot\nabla\ell^\theta), h^\theta\rangle_{L^2}.\end{aligned}$$

We remark that (see ref. 5)

$$\begin{aligned}|\langle P_h B(\ell^u, \ell^u), h^u\rangle_{L^2}| &\leqslant C|\Lambda h^u|_{L^2}|\Lambda\ell^u|^2_{L^2}, \\ |\langle P_h B(h^u, \ell^u), h^u\rangle_{L^2}| &\leqslant C|\Lambda h^u|_{L^2}|h^u|_{L^2}|\Lambda\ell^u|_{L^2}, \\ |\langle P_h(h^u\cdot\nabla\ell^\theta), h^\theta\rangle_{L^2}| &\leqslant C|\Lambda h^u|_{L^2}|\nabla^2\ell^\theta|_{L^2}|h^\theta|_{L^2}, \\ |\langle P_h(\ell^u\cdot\nabla\ell^\theta), h^\theta\rangle_{L^2}| &\leqslant C|\Lambda h^\theta|_{L^2}|\nabla\ell^\theta|_{L^2}|\Lambda\ell^u|_{L^2}\end{aligned}$$

and

$$|\ \langle\sigma a(h^\theta), h^u\rangle_{L^2}| \leqslant \frac{\nu}{2}|h^u|^2_{L^2} + \frac{C\sigma^2}{2\nu}|h^\theta|^2_{L^2}$$

Using the above inequalities, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(|h^u(t)|^2_{L^2} + \frac{C\sigma^2}{\kappa\nu}|h^\theta(t)|^2_{L^2}\right) \\ &\quad\leqslant -\frac{\nu}{2}|\Lambda h^u|^2_{L^2} + C|\Lambda h^u|_{L^2}|h^u|_{L^2}|\Lambda\ell^u|_{L^2} + C|\Lambda h^u|_{L^2}|\Lambda\ell^u|^2_{L^2} \\ &\qquad -\frac{C\sigma^2}{2\nu}|\nabla h^\theta|^2_{L^2} + C|\Lambda h^u|_{L^2}|\nabla^2\ell^\theta|_{L^2}|h^\theta|_{L^2} + C|\Lambda h^\theta|_{L^2}|\nabla\ell^\theta|_{L^2}|\Lambda\ell^u|_{L^2} \\ &\quad\leqslant C|h^u|^2_{L^2}|\Lambda\ell^u|^2_{L^2} + C|\Lambda\ell^u|^4_{L^2} + C|h^\theta|^2_{L^2}|\nabla^2\ell^\theta|^2_{L^2} + C|\nabla\ell^\theta|^4_{L^2}.\end{aligned}$$

Since ℓ^u, $\ell^\theta \in L^2_\ell$, we have $|\Lambda \ell^u|_{L^2} \leqslant (N^+)|\ell^u|_{L^2}$ and $|\nabla^2 \ell^\theta|_{L^2} \leqslant (N^+)^2 |\ell^\theta|_{L^2}$. Using Gronwall's lemma, we have

$$
\begin{aligned}
&|h^u(t)|^2_{L^2} + \frac{\sigma^2}{\kappa\nu}|h^\theta(t)|^2_{L^2} \\
&\quad \leqslant C_1(|h^u(0)|^2_{L^2} + \frac{\sigma^2}{\kappa\nu}|h^\theta(0)|^2_{L^2}) \exp\left(C_3 \int_0^t |\ell^u|^2_{L^2} ds + C_4 \int_0^t |\ell^\theta|^2_{L^2}\, ds\right) \\
&\qquad + C_2\left(\int_0^t |\ell^u|^4_{L^2}\, ds + \int_0^t |\ell^\theta|^4_{L^2}\, ds\right) \\
&\qquad \times \exp\left(C_3 \int_0^t |\ell^u|^2_{L^2} ds + C_4 \int_0^t |\ell^\theta|^2_{L^2} ds\right),
\end{aligned}
$$

for some constants C_1, C_2, C_3 and C_4. ∎

Fix $L \in \mathcal{P}$. We consider the following ODEs for comparing the process $l(t)$ to the associated Galerkin approximation living on $\mathbb{L}^2_l$ which we will denote by $x(t)$. Take $x(t)$, $y(t)$ as the solution defined by the following stochastic ODEs.

$$
\begin{aligned}
&dx(t) = [-\nu\Lambda^2 x + P_\ell B(x, x) + \sigma a(y)]dt + dW_u(t), \\
&dy(t) = [\kappa\Delta y + P_\ell(x \cdot \nabla y)]\, dt + dW_\theta(t), \\
&x(0) = \ell^u(0), \qquad y(0) = \ell^\theta(0).
\end{aligned}
$$

We do not compare $x(t)$, $y(t)$ directly to $\ell^u(t)$, $\ell^\theta(t)$ but instead to a modified version of $\ell^u(t)$, $\ell^\theta(t)$, which will be denoted by $z_1(t)$ and $z_2(t)$.

$$
\begin{aligned}
dz_1(t) &= [-\nu\Lambda^2 z_1 + P_\ell B(z_1, z_1) + \sigma a(z_2) + \Theta_t(Z^t)G_1(z_1, \Phi^u_t(Z^t_1, h^u_0))]\, dt \\
&\qquad + dW_u(t), \\
dz_2(t) &= [\kappa\Delta z_2 + P_\ell(z_1 \cdot \nabla z_2) + \Theta_t(Z^t)G_2(z_2, \Phi_t(Z^t, h_0))]\, dt + dW_\theta(t), \\
z_1(0) &= \ell^u(0), \qquad z_2(0) = \ell^\theta(0),
\end{aligned}
$$

where $Z^t = (Z^t_1, Z^t_2)$, $h^u_0 = \Phi^u_0(L)$ and $h^\theta_0 = \Phi^\theta_0(L)$. For any fixed $b_0 > 1$, we define

$$
\Theta_s(Z^s) = \begin{cases} 1 & \text{if } \int_0^s |Z^s(r)|^4_{L^2}\, dr < (b_0 C_0)^4 T, \\ 0 & \text{otherwise.} \end{cases}
$$

For any $B \subset \mathbb{L}^2_\ell$, define

$$
[B] = \{(u, \theta) \in C([0, t], \mathbb{L}^2_\ell) : (u(t), \theta(t)) \in B\}.
$$

Then $R_t(L(0), B) = Q_t(L, [B])$. Letting $Q_t^{x,y}(L(0), \cdot)$ and $Q_t^z(L, \cdot)$ be the two measures induced on $C([0, t], \mathbb{L}_\ell^2)$ by the dynamics of x and z, respectively.

Lemma 3.5. Fix any $b_0 > 1$. Then for any $L^0 \in \mathcal{P}$, $R_t(L^0, \cdot)$ is equivalent to the Lebesgue measure $m(\cdot)$.

For any μ, let $P_\ell \mu$ be its projection to the low mode space, i.e. $(P_\ell \mu)(B) = \mu(P_\ell^{-1}(B))$. Then we have the following direct consequence of Lemma 3.5.

Corollary 3.6. If μ is an ergodic invariant measure then $P_\ell \mu$ has a component which is equivalent to the Lebesgue measure.

Proof of Lemma 3.5. Let $Q_t^{x,y}(L^0, \cdot)$ and $Q_t^\ell(L^0, \cdot)$ be the two measures induced on $C([0, t], \mathbb{L}_\ell^2)$ by the dynamics of (x, y) and ℓ, respectively. Observe that $z(t) = \ell(t)$ as long as the trajectories stay in A_T, the Girsanov theorem will imply $Q_t^{x,y}(L^0, A_T)$ is equivalent to $Q_t^\ell(L^0, A_T)$ for $0 \leqslant t \leqslant T$ if the following Novikov condition holds:

$$\mathbb{E} \exp\left\{\frac{1}{2}\int_0^t |\Sigma^{-1}\Theta_s(Z^s)|^2 |G(z(s), \Phi_s(z(s), \Phi_s(Z^s, h_0))|_{L^2}^2 ds\right\} < \infty,$$

where

$$G(\vec{a}, \vec{b}) = G\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) = \begin{pmatrix} G_1(a_1, b_1) \\ G_2(\vec{a}, \vec{b}) \end{pmatrix}.$$

We will prove the stronger condition

$$\sup_{z(\cdot) \in A_T} \int_0^t |G(z(s), \Phi_s(Z^s, h_0))|_{L^2}^2 \, ds < \infty.$$

We obtain the following estimate on G(see ref. 6):

$$|G(z(s), \Phi_s(Z^s, h_0))|_{L^2}^2 \leqslant C'[|z(s)|_{L^2}^2 |h(s)|_{L^2}^2 + |h(s)|_{L^2}^4],$$

where $h(s) = \Phi_s(Z^s, h_0)$. We note that

$$|z(s)|_{L^2}^2 \leqslant |z_1(s)|_{L^2}^2 + |z_2(s)|_{L^2}^2, \quad |h(s)|_{L^2}^2 \leqslant |h^u(s)|_{L^2}^2 + |h^\theta(s)|_{L^2}^2.$$

By Lemma 3.4, we know that if z is in A, then we know that

$$\sup_{s\in[0,t]} |h(t)|_{L^2}^2 \leqslant \sup|h^u(t)|_{L^2}^2 + \sup|h^\theta(t)|_{L^2}^2 \leqslant C_2(|h_0|_{L^2}, b_0, C_0, T).$$

Hence for any $z \in A_T$, we have

$$\begin{aligned}
&\int_0^t |G(z(s), \Phi_s(Z^s, h_0))|_{L^2}^2\, ds \\
&\quad\leqslant C' \int_0^t [|z(s)|_{L^2}^2 |h(s)|_{L^2}^2 + |h(s)|_{L^2}^4]\, ds \\
&\quad\leqslant C' \left(\int_0^t |z(s)|_{L^2}^4 ds\right)^{1/2} \left(\int_0^t |h(s)|_{L^2}^4\, ds\right)^{1/2} + C' C_2^4 t \\
&\quad\leqslant C'(b_0C_0)^2 T^{1/2} C_2^2 t^{1/2} + C' C_2^4 t.
\end{aligned}$$

Hence Novikov's condition holds and the lemma is proved. ∎

Proof of Theorem 1. The proof is the same as the proof of refs. 6 and 11. We just provide the sketch of the proof. For two different ergodic stationary measure μ_1, μ_2, we can extend them to $\mu_{1,p}$ and $\mu_{2,p}$ onto the path space $\mathcal{P}$. We can find a functional F defined as above such that $\int F(L)\, d\mu_{1,p}(L) \neq \int F(L)\, d\mu_{2,p}(L)$ Using Fubini's theorem, we can find a set A_i such that $\mu_{i,p}(\mathcal{P}^i(\ell)|\ell) = 1$ for all $\ell \in A_i$ and $P_\ell \mu_i(A_i) = 1$. Define $A = A_1 \cap A_2$. By Corollary 3.6, there exists some $\ell^* \in A$. Thus there exist $L_{*,1} \in \mathcal{P}^1(\ell^*)$ and $L_{*,2} \in \mathcal{P}^2(\ell^*)$. By Lemma 3.3, $Q_\infty(L_{*,1}, \cdot)$ and $Q_\infty(L_{*,2}, \cdot)$ are equivalent. Hence we can find $B_i \subset C([0, \infty), \mathbb{L}^2)$ such that the time average of F converges to $\bar{F}_i$ for all futures in B_i and $Q_\infty(L_{*,i}, B_i) = 1$ for $i = 1, 2$. Since $B_1 \cap B_2$ is nonempty, this contradicts the assumption. This completes the proof. ∎

4. REGULARITY OF THE TRANSTION DENSITY IN THE CASE OF GALERKIN TRUNCATION

In the following, we denote the vector-valued index by g, h, k–n, and the scalar-valued index by i, j. In this part, we consider the finite dimensional Galerkin approximations of the three-dimensional reactive Boussinesq equations with degenerate stochastical forcing. in the domain $\mathbb{T}^3$, with periodic boundary conditions as stated in the Section 1.

$$du_k = [-\nu|k|^2 u_k - i\sum(k\cdot u_h)(u_l - \frac{k\cdot u_l}{|k|^2}k)$$
$$-\sigma(\tilde{\theta}_k - \frac{k\cdot\tilde{\theta}_k}{|k|^2}k)]\,dt + q_{uk}d\beta^k_{ut}, \tag{4.1}$$
$$d\theta_k = [-\kappa|k|^2\theta_k - i\sum(k\cdot u_h)\theta_l]\,dt + q_{\theta k}\,d\beta^k_{\theta t}, \tag{4.2}$$

where the sum is over $h,\ l\in\mathcal{K}_N$ and $h+l=k$.

We set $u_k=(r_k^j+is_k^j)$, $\theta_k=\tilde{r}_k+i\tilde{s}_k$, and $\tilde{\theta}_k=\vec{r}_k+i\vec{s}_k=(0,\,0,\,\tilde{r}_k)^T+i(0,\,0,\,\tilde{s}_k)^T$, where $k\cdot r_k=k\cdot s_k=0$ and $r_k^j,\ s_k^j,\ \tilde{r}_k$, and $\tilde{s}_k$ are real-valued. Let $q_{uk}=q^r_{uk}+iq^s_{uk}$ and $q_{\theta k}=q^r_{\theta k}+iq^s_{\theta k}$. Since $u_{-k}=\bar{u}_k$ and $\theta_{-k}=\bar{\theta}_k$, we only need to consider a smaller set of indices $k\in\mathcal{K}_N$. We set

$$\mathcal{K}^1_N = \{k\in\mathbb{Z}^3|\ |k|_\infty\leqslant N,\ k_3>0\},$$
$$\mathcal{K}^2_N = \{k\in\mathbb{Z}^3|\ |k|_\infty\leqslant N,\ k_3=0,\ k_2>0\},$$
$$\mathcal{K}^3_N = \{k\in\mathbb{Z}^3|\ |k|_\infty\leqslant N,\ k_3=k_2=0,\ k_1>0\}$$

and $\tilde{\mathcal{K}}=\mathcal{K}^1_N\cup\mathcal{K}^2_N\cup\mathcal{K}^3_N$ in such a way that $\mathcal{K}_N=\tilde{\mathcal{K}}\cup(-\tilde{\mathcal{K}})$ and $\tilde{\mathcal{K}}\cap(-\tilde{\mathcal{K}})=\phi$. Notice that number of the elements of $\tilde{\mathcal{K}}=(1/2)[(2N+1)^3-1]$, we call such number D. Now if $k\in\tilde{\mathcal{K}}$, we can write

$$\sum_{\substack{h+l=k\\ h,\,l\in\mathcal{K}_N}} = \sum_{\substack{h+l=k\\ h,\,l\in\tilde{\mathcal{K}}}} + \sum_{\substack{h-l=k\\ h,\,l\in\tilde{\mathcal{K}}}} + \sum_{\substack{l-h=k\\ h,\,l\in-\tilde{\mathcal{K}}.}}$$

We denote by $\sum^*$ the sum when indices in $\tilde{\mathcal{K}}$.

So we have

$$du_k + [\nu|k|^2u_k + i\sum^*_{h+l=k}(k\cdot u_h)(u_l-\frac{k\cdot u_l}{|k|^2}k) + i\sum^*_{h-l=k}(k\cdot u_h)(\bar{u}_l-\frac{k\cdot\bar{u}_l}{|k|^2}k)$$
$$+i\sum^*_{l-h=k}(k\cdot\bar{u}_h)(u_l-\frac{k\cdot u_l}{|k|^2}k)+\sigma(\tilde{\theta}_k-\frac{k\cdot\tilde{\theta}_k}{|k|^2}k)]\,dt = q_{uk}\,d\beta^k_{ut}$$

and

$$d\theta_k + [\kappa|k|^2\theta_k + i\sum^*_{h+l=k}(k\cdot u_h)\theta_l + i\sum^*_{h-l=k}(k\cdot u_h)\bar{\theta}_l$$
$$+i\sum^*_{l-h=k}(k\cdot\bar{u}_h)\theta_l]dt = q_{\theta k}\,d\beta^k_{\theta t}.$$

By splitting into the real part and imaginary part, we obtain

$$\begin{aligned}
dr_k + \Bigg[\nu|k|^2 r_k &- \sum_{h+l=k}^{*} (k\cdot r_h)(s_l - \frac{k\cdot s_l}{|k|^2}k) + (k\cdot s_h)(r_l - \frac{k\cdot r_l}{|k|^2}k) \\
&+ \sum_{h-l=k}^{*} (k\cdot r_h)(s_l - \frac{k\cdot s_l}{|k|^2}k) - (k\cdot s_h)(r_l - \frac{k\cdot r_l}{|k|^2}k) \\
&- \sum_{l-h=k}^{*} (k\cdot r_h)(s_l - \frac{k\cdot s_l}{|k|^2}k) - (k\cdot s_h)(r_l - \frac{k\cdot r_l}{|k|^2}k) \\
&+ \sigma\left(\vec{r}_k - \frac{k\cdot\vec{r}_k}{|k|^2}k\right)\Bigg] dt = q^r_{uk}\, d\beta^k_{ut}
\end{aligned}$$

and

$$\begin{aligned}
ds_k + \Bigg[\nu|k|^2 s_k &+ \sum_{h+l=k}^{*} (k\cdot r_l)(r_l - \frac{k\cdot r_l}{|k|^2}k) - (k\cdot s_h)(s_l - \frac{k\cdot s_l}{|k|^2}k) \\
&+ \sum_{h-l=k}^{*} (k\cdot r_l)(r_l - \frac{k\cdot r_l}{|k|^2}k) + (k\cdot s_h)(s_l - \frac{k\cdot s_l}{|k|^2}k) \\
&+ \sum_{l-h=k}^{*} (k\cdot r_h)(r_l - \frac{k\cdot r_l}{|k|^2}k) + (k\cdot s_h)(s_l - \frac{k\cdot s_l}{|k|^2}k) \\
&+ \sigma\left(\vec{s}_k - \frac{k\cdot\vec{s}_k}{|k|^2}k\right)\Bigg] dt = q^s_{uk}\, d\beta^k_{ut},
\end{aligned}$$

$$\begin{aligned}
d\tilde{r}_k + [\kappa|k|^2\tilde{r}_k &- \sum_{h+l=k}^{*} (k\cdot r_h)\tilde{s}_l + (k\cdot s_h)\tilde{r}_l + \sum_{h-l=k}^{*} (k\cdot r_h)\tilde{s}_l - (k\cdot s_h)\tilde{r}_l \\
&- \sum_{l-h=k}^{*} (k\cdot r_h)\tilde{s}_l - (k\cdot s_h)\tilde{r}_l]\, dt = q^r_{\theta k}\, d\beta^k_{\theta t}
\end{aligned}$$

and

$$\begin{aligned}
d\tilde{s}_k + [\kappa|k|^2\tilde{s}_k &+ \sum_{h+l=k}^{*} (k\cdot r_h)\tilde{r}_l - (k\cdot s_h)\tilde{s}_l \\
&+ \sum_{h-l=k}^{*} (k\cdot r_h)\tilde{r}_l + (k\cdot s_h)\tilde{s}_l \\
&+ \sum_{l-h=k}^{*} (k\cdot r_h)\tilde{r}_l + (k\cdot s_h)\tilde{s}_l]\, dt = q^s_{\theta k}\, d\beta^k_{\theta t}.
\end{aligned}$$

We let

$$\begin{aligned}
F_{r_k^i} &= -\nu|k|^2 r_k^i + \sum_{h+l=k}^{*} (k\cdot r_h)(s_l^i - \frac{k\cdot s_l}{|k|^2}k_i) + (k\cdot s_h)(r_l^i - \frac{k\cdot r_l}{|k|^2}k_i)\\
&\quad - \sum_{h-l=k}^{*} (k\cdot r_h)(s_l^i - \frac{k\cdot s_l}{|k|^2}k_i) - (k\cdot s_h)(r_l^i - \frac{k\cdot r_l}{|k|^2}k_i)\\
&\quad + \sum_{l-h=k}^{*} (k\cdot r_h)(s_l^i - \frac{k\cdot s_l}{|k|^2}k_i) - (k\cdot s_h)(r_l^i - \frac{k\cdot r_l}{|k|^2}k_i) + \sigma\delta_{i3}\tilde{r}_k,\\
F_{s_k^i} &= -\nu|k|^2 s_k^i - \sum_{h+l=k}^{*} (k\cdot r_h)(r_l^i - \frac{k\cdot r_l}{|k|^2}k_i) - (k\cdot s_h)(s_l^i - \frac{k\cdot s_l}{|k|^2}k_i)\\
&\quad - \sum_{h-l=k}^{*} (k\cdot r_h)(r_l^i - \frac{k\cdot r_l}{|k|^2}k_i) + (k\cdot s_h)(s_l^i - \frac{k\cdot s_l}{|k|^2}k_i)\\
&\quad - \sum_{l-h=k}^{*} (k\cdot r_h)(r_l^i - \frac{k\cdot r_l}{|k|^2}k_i) + (k\cdot s_h)(s_l^i - \frac{k\cdot s_l}{|k|^2}k_i) + \sigma\delta_{i3}\tilde{s}_k,\\
\tilde{F}_{\tilde{r}_k} &= -\kappa|k|^2\tilde{r}_k + \sum_{h+l=k}^{*} (k\cdot r_h)\tilde{s}_l + (k\cdot s_h)\tilde{r}_l\\
&\quad - \sum_{h-l=k}^{*} (k\cdot r_h)\tilde{s}_l - (k\cdot s_h)\tilde{r}_l\\
&\quad + \sum_{l-h=k}^{*} (k\cdot r_h)\tilde{s}_l - (k\cdot s_h)\tilde{r}_l,\\
\tilde{F}_{\tilde{s}_k} &= -\kappa|k|^2\tilde{s}_k - \sum_{h+l=k}^{*} (k\cdot r_h)\tilde{r}_l - (k\cdot s_h)\tilde{s}_l\\
&\quad - \sum_{h-l=k}^{*} (k\cdot r_h)\tilde{r}_l + (k\cdot s_h)\tilde{s}_l\\
&\quad - \sum_{l-h=k}^{*} (k\cdot r_h)\tilde{r}_l + (k\cdot s_h)\tilde{s}_l.
\end{aligned}$$

The solutions $(r(t), \tilde{r}(t), s(t), \tilde{s}(t))$ of our equations is a Markov process whose state space is a linear subspace U of $\mathbb{R}^{8D}$, where $D = \#\tilde{\mathcal{K}}$. We can write $U = \oplus_{k\in\tilde{\mathcal{K}}}(R_k \oplus \tilde{R}_k \oplus S_k \oplus \tilde{S}_k)$, where (in the below $h \neq k$)

$$\begin{aligned}
R_k &= \{(r,\, \tilde{r},\, s,\, \tilde{s}) \in \mathbb{R}^{8D} | r_k \cdot k = \tilde{r}_k = s_k = \tilde{s}_k = r_h = \tilde{r}_h = s_h = \tilde{s}_h = 0\},\\
S_k &= \{(r,\, \tilde{r},\, s,\, \tilde{s}) \in \mathbb{R}^{8D} | s_k \cdot k = r_k = \tilde{r}_k = \tilde{s}_k = r_h = \tilde{r}_h = s_h = \tilde{s}_h = 0\},\\
\tilde{R}_k &= \{(r,\, \tilde{r},\, s,\, \tilde{s}) \in \mathbb{R}^{8D} |\, r_k = s_k = \tilde{s}_k = 0, r_h = \tilde{r}_h = s_h = \tilde{s}_h = 0\},\\
\tilde{S}_k &= \{(r,\, \tilde{r},\, s,\, \tilde{s}) \in \mathbb{R}^{8D} |\, r_k = \tilde{r}_k = s_k = 0, r_h = \tilde{r}_h = s_h = \tilde{s}_h = 0\}.
\end{aligned}$$

Then we define the Lie algebra $\mathcal{U}$ corresponding to the state space U of the solution process as

$$\begin{aligned}
\mathcal{U} = \Big\{ G | G = \sum_{k \in \tilde{\mathcal{K}}} G_{r_{k_i}} \frac{\partial}{\partial r_k^i} + \tilde{G}_{\tilde{r}_k} \frac{\partial}{\partial \tilde{r}_k} + G_{s_k^i} \frac{\partial}{\partial s_k^i} + \tilde{G}_{\tilde{s}_k} \frac{\partial}{\partial \tilde{s}_k}\\
\text{and } k \cdot G_{r_k^i} = k \cdot G_{s_k^i} = 0 \Big\}.
\end{aligned}$$

We define also the subspaces $\mathcal{U}_k = \mathcal{R}_k \oplus \tilde{\mathcal{R}}_k \oplus \mathcal{G}_k \oplus \tilde{\mathcal{G}}_k$ of U of constant vector fields, where

$$\begin{aligned}
&\mathcal{R}_k = \left\{ \sum_{i=1,2,3} r_k^i \frac{\partial}{\partial r_k^i} | r_k \cdot k = 0 \right\}, \qquad \mathcal{G}_k = \left\{ \sum_{i=1,2,3} s_k^i \frac{\partial}{\partial s_k^i} | s_k \cdot k = 0 \right\},\\
&\tilde{\mathcal{R}}_k = \left\{ \tilde{r}_k \frac{\partial}{\partial \tilde{r}_k} \right\}, \quad \text{and} \quad \tilde{\mathcal{G}}_k = \left\{ \tilde{s}_k \frac{\partial}{\partial \tilde{s}_k} \right\}.
\end{aligned}$$

We wish to find the reasonable conditions on the set $\mathcal{N}$ of forced modes in such a way that the algebra generated by the fields $\{F_0\} \cup \mathcal{U}_k$, $k \in \mathcal{N}$, where $F_0 = \sum_{k \in \tilde{\mathcal{K}},\, i=1,2,3} F_{r_k^i} \frac{\partial}{\partial r_k^i} + F_{s_k^i} \frac{\partial}{\partial s_k^i} + \tilde{F}_{\tilde{r}_k} \frac{\partial}{\partial \tilde{r}_k} + \tilde{F}_{\tilde{s}_k} \frac{\partial}{\partial \tilde{s}_k}$, contains all the constant vector fields of $\mathcal{U}$. So the generated Lie algebra if evaluated at each point of $\mathcal{U}$ gives $\mathcal{U}$ itself and we obtain the desired hypoelliptic property. As indicated in Section 1, we reduced the proof of the uniqueness of the invariant measure to two parts. The irreducibiblity is proved in the next section. In this section, we prove the transition probability densities are regular, i.e., strongly Feller. To show the strongly Feller property of the Galerkin approximations of three-dimensional Boussinesq equations, we use the Hörmander's Theorem. Thus we need to compute the Lie bracket of the form $[[F_0,\, V], W]$ for general constant vector field V and W. The following lemma can be obtained from the direct computations.

Lemma 4.1. Let $m,\, n \in \tilde{\mathcal{K}}$ and $V \in \mathcal{U}_m$, $W \in \mathcal{U}_n$, with

$$V = \sum_{j=1}^{3} \left(v_j^r \frac{\partial}{\partial r_m^j} + v_j^s \frac{\partial}{\partial s_m^j} \right) + \tilde{v}^r \frac{\partial}{\partial \tilde{r}_m} + \tilde{v}^s \frac{\partial}{\partial \tilde{s}_m}$$

and

$$W=\sum_{j=1}^{3}\left(w_j^r\frac{\partial}{\partial r_n^j}+w_j^s\frac{\partial}{\partial s_n^j}\right)+\tilde{w}^r\frac{\partial}{\partial\tilde{r}_n}+\tilde{w}^s\frac{\partial}{\partial\tilde{s}_n}.$$

If $k=m+n$, $h=n-m$, and $g=m-n$, then

$$\begin{aligned}
&[[F_0,\ V],W]\\
&\quad=[(v^r\cdot k)P_k(w^s)+(w^s\cdot k)P_k(v^r)+(v^s\cdot k)P_k(w^r)+(w^r\cdot k)P_k(v^s)]\cdot\frac{\partial}{\partial r_k}\\
&\qquad+[(v^s\cdot k)P_k(w^s)+(w^s\cdot k)P_k(v^s)-(v^r\cdot k)P_k(w^r)-(w^r\cdot k)P_k(v^r)]\cdot\frac{\partial}{\partial s_k}\\
&\qquad+[(v^r\cdot h)P_h(w^s)+(w^s\cdot h)P_h(v^r)-(v^s\cdot h)P_h(w^r)-(w^r\cdot h)P_h(v^s)]\cdot\frac{\partial}{\partial r_h}\\
&\qquad-[(v^r\cdot h)P_g(w^r)+(w^r\cdot h)P_h(v^r)+(v^s\cdot h)P_h(w^s)+(w^s\cdot h)P_h(v^s)]\cdot\frac{\partial}{\partial s_h}\\
&\qquad+[(v^s\cdot g)P_g(w^r)+(w^r\cdot g)P_g(v^s)-(v^r\cdot g)P_g(w^s)-(w^s\cdot g)P_g(v^r)]\cdot\frac{\partial}{\partial r_g}\\
&\qquad-[(v^r\cdot g)P_g(w^r)+(w^r\cdot g)P_g(v^r)+(v^s\cdot g)P_g(w^s)+(w^s\cdot g)P_g(v^s)]\cdot\frac{\partial}{\partial s_g}\\
&\qquad+[(w^s\cdot k)\tilde{v}^r+(w^r\cdot k)\tilde{v}^s+(v^r\cdot k)\tilde{w}^s+(v^s\cdot k)\tilde{w}^r]\frac{\partial}{\partial\tilde{r}_k}\\
&\qquad+[(w^s\cdot h)\tilde{v}^r-(w^r\cdot h)\tilde{v}^s+(v^r\cdot h)\tilde{w}^s-(v^s\cdot h)\tilde{w}^r]\frac{\partial}{\partial\tilde{r}_h}\\
&\qquad+[-(w^s\cdot g)\tilde{v}^r+(w^r\cdot g)\tilde{v}^s-(v^r\cdot g)\tilde{w}^s+(v^s\cdot g)\tilde{w}^r]\frac{\partial}{\partial\tilde{r}_g}\\
&\qquad+[-(w^r\cdot k)\tilde{v}^r-(v^r\cdot k)\tilde{w}^r+(w^s\cdot k)\tilde{v}^s+(v^s\cdot k)\tilde{w}^s]\frac{\partial}{\partial\tilde{s}_k}\\
&\qquad-[(w^r\cdot h)\tilde{v}^r+(v^r\cdot h)\tilde{w}^r+(w^s\cdot h)\tilde{v}^s+(v^s\cdot h)\tilde{w}^s]\frac{\partial}{\partial\tilde{s}_h}\\
&\qquad-[(w^r\cdot g)\tilde{v}^r+(v^r\cdot g)\tilde{w}^r+(w^s\cdot g)\tilde{v}^s+(v^s\cdot g)\tilde{w}^s]\frac{\partial}{\partial\tilde{s}_g},
\end{aligned}$$

where P_k is the projection of $\mathbb{R}^3$ on the plane orthogonal to the vector k, and in the above formula the terms corresponding to indices out of $\tilde{\mathcal{K}}$ are zero.

Proof. We compute the derivatives of the components of F_0

$$\frac{\partial F_{r_k^i}}{\partial r_m^j} = -\nu|k|^2\delta_{ij}\delta_{km} + k_j(s_{k-m}^i - s_{m-k}^i + s_{m+k}^i) + k\cdot(s_{k-m} - s_{m-k} + s_{k+m})\left(\delta_{ij} - \frac{2k_ik_j}{|k|^2}\right),$$

$$\frac{\partial F_{s_k^i}}{\partial r_m^j} = k_j(r_{k-m}^i + r_{m-k}^i - r_{m+k}^i) + k\cdot(r_{k-m} + r_{m-k} - r_{m+k})\left(\delta_{ij} - \frac{2k_ik_j}{|k|^2}\right),$$

$$\frac{\partial F_{s_k^i}}{\partial r_m^j} = -k_j(r_{k-m}^i + r_{m-k}^i + r_{m+k}^i) - k\cdot(r_{k-m} + r_{m-k} + r_{m+k})\left(\delta_{ij} - \frac{2k_ik_j}{|k|^2}\right),$$

$$\frac{\partial F_{s_k^i}}{\partial s_m^j} = -\nu|k|^2\delta_{ij}\delta_{km} + k_j(s_{k-m}^i - s_{m-k}^i - s_{m+k}^i) + k\cdot(s_{k-m} - s_{m-k} - s_{m+k})\left(\delta_{ij} - \frac{2k_ik_j}{|k|^2}\right),$$

$$\frac{\partial F_{r_k^j}}{\partial \tilde r_m} = \frac{\partial F_{s_k^j}}{\partial \tilde s_m} = \sigma\delta_{j3}, \qquad \frac{\partial F_{r_k^j}}{\partial \tilde s_m} = \frac{\partial F_{s_k^j}}{\partial \tilde r_m} = 0,$$

$$\frac{\partial \tilde F_{\tilde r_k}}{\partial \tilde r_m} = -\kappa|k|^2\delta_{km} + k\cdot(s_{k-m} - s_{m-k} + s_{m+k}),$$

$$\frac{\partial \tilde F_{\tilde r_k}}{\partial \tilde s_m} = k\cdot(r_{k-m} + r_{m-k} - r_{m+k}),$$

$$\frac{\partial \tilde F_{\tilde s_k}}{\partial \tilde r_m} = -k\cdot(r_{k-m} + r_{m-k} + r_{m+k}),$$

$$\frac{\partial \tilde F_{\tilde s_k}}{\partial \tilde s_m} = -\kappa|k|^2\delta_{km} + k\cdot(s_{k-m} - s_{m-k} - s_{m+k}),$$

$$\frac{\partial \tilde F_{\tilde r_k}}{\partial r_m^j} = k_j(\tilde s_{k-m} - \tilde s_{m-k} + \tilde s_{m+k}),$$

$$\frac{\partial \tilde F_{\tilde r_k}}{\partial s_m^j} = k_j(\tilde r_{k-m} + \tilde r_{m-k} - \tilde r_{m+k}),$$

$$\frac{\partial \tilde F_{\tilde s_k}}{\partial s_m^j} = k_j(\tilde s_{k-m} - \tilde s_{m-k} - \tilde s_{m+k})$$

and

$$\frac{\partial \tilde{F}_{\tilde{s}_k}}{\partial r_m^j} = -k_j(\tilde{r}_{k-m} + \tilde{r}_{m-k} - \tilde{r}_{m+k}),$$

Thus the nonvanishing second derivatives are the following(we set $A^i_{jl}(k) = \delta_{il}k_j + \delta_{ij}k_l - 2(k_i k_j k_l)/|k|^2$.)

$$\begin{aligned}
\frac{\partial^2 F_{r_k^i}}{\partial s_n^l\, \partial r_m^j} &= (\delta_{n,k-m} - \delta_{n,m-k} + \delta_{n,m+k})A^i_{j,l}(k),\\
\frac{\partial^2 F_{s_k^i}}{\partial r_n^l\, \partial r_m^j} &= -(\delta_{n,k-m} + \delta_{n,m-k} + \delta_{n,m+k})A^i_{j,l}(k),\\
\frac{\partial^2 F_{s_k^i}}{\partial s_n^l\, \partial s_m^j} &= (\delta_{n,k-m} - \delta_{n,m-k} - \delta_{n,m+k})A^i_{jl}(k),\\
\frac{\partial^2 \tilde{F}_{\tilde{r}_k}}{\partial \tilde{r}_m\, \partial s_n^l} &= k_l(\delta_{n,k-m} - \delta_{n,m-k} + \delta_{n,m+k}),\\
\frac{\partial^2 \tilde{F}_{\tilde{r}_k}}{\partial \tilde{s}_m\, \partial r_n^l} &= k_l(\delta_{n,k-m} - \delta_{n,m-k} + \delta_{n,m+k})\\
\frac{\partial^2 \tilde{F}_{\tilde{s}_k}}{\partial \tilde{s}_n \partial s_m^j} &= k_j(\delta_{n,k-m} - \delta_{n,m-k} - \delta_{n,m+k})
\end{aligned}$$

and

$$\frac{\partial^2 \tilde{F}_{\tilde{s}_k}}{\partial \tilde{r}_n\, \partial r_m^j} = -k_j(\delta_{n,k-m} + \delta_{n,m-k} + \delta_{n,m+k}).$$

Computing the bracket produces

$$\begin{aligned}
&[[F_0,\, V],\, W]\\
&\quad = \sum_{k\in\tilde{\mathcal{K}}} \sum_{i,j,l=1}^{3} \left\{ \left(v_j^s w_l^r \frac{\partial^2 F_{r_k^i}}{\partial s_m^j\, \partial r_n^l} + v_j^r w_l^s \frac{\partial^2 F_{r_k^i}}{\partial r_m^j\, \partial s_n^l} \right) \frac{\partial}{\partial r_k^i} \right.\\
&\qquad \left. + \left(v_j^r w_l^r \frac{\partial^2 F_{s_k^i}}{\partial r_m^j\, \partial r_n^l} + v_j^s w_l^s \frac{\partial^2 F_{s_k^i}}{\partial s_m^j \partial s_n^l} \right) \frac{\partial}{\partial s_k^i} \right\}
\end{aligned}$$

$$+\sum_{k\in\tilde{\mathcal{K}}}\sum_{j=1}^{3}\Bigg\{\Bigg(\tilde{v}^r w_j^s\frac{\partial^2\tilde{F}_{\tilde{r}_k}}{\partial\tilde{r}_m\,\partial s_n^j}+v_j^r\tilde{w}^s\frac{\partial^2\tilde{F}_{\tilde{r}_k}}{\partial\tilde{s}_n\,\partial r_m^j}$$
$$+\tilde{w}^r v_j^s\frac{\partial^2\tilde{F}_{\tilde{r}_k}}{\partial\tilde{r}_n\,\partial s_m^j}+\tilde{v}^s w_j^r\frac{\partial^2\tilde{F}_{\tilde{r}_k}}{\partial\tilde{s}_m\,\partial r_n^j}\Bigg)\frac{\partial}{\partial\tilde{r}_k}$$
$$+\Bigg(\tilde{v}^r w_j^r\frac{\partial^2\tilde{F}_{\tilde{s}_k}}{\partial\tilde{r}_m\,\partial r_n^j}+v_j^r\tilde{w}^r\frac{\partial^2\tilde{F}_{\tilde{s}_k}}{\partial\tilde{r}_n\,\partial r_m^j}+\tilde{v}^s w_j^s\frac{\partial^2\tilde{F}_{\tilde{s}_k}}{\partial\tilde{s}_m\,\partial s_n^j}+v_j^s\tilde{w}^s\frac{\partial^2\tilde{F}_{\tilde{s}_k}}{\partial\tilde{s}_n\,\partial s_m^j}\Bigg)\frac{\partial}{\partial\tilde{s}_k}\Bigg\}.$$

We analyze the coefficients of the $\partial_{r_k^i}$ and $\partial_{s_k^i}$

$$\begin{aligned}
&v_j^s w_l^r\frac{\partial^2 F_{r_k^i}}{\partial s_m^j\,\partial r_n^l}+v_j^r w_l^s\frac{\partial^2 F_{r_k^i}}{\partial r_m^j\,\partial s_n^l}\\
&\quad=(\delta_{m,k-n}-\delta_{m,n-k}+\delta_{m,n+k})[(v^s\cdot k)P_k(w^r)_i+(w^r\cdot k)P_k(v^s)_i]\\
&\qquad+(\delta_{n,k-m}-\delta_{n,m-k}+\delta_{n,m+k})[(v^r\cdot k)P_k(w^s)_i+(w^s\cdot k)P_k(v^r)_i],\\
&v_j^r w_l^r\frac{\partial^2 F_{s_k^i}}{\partial r_m^j\,\partial_n^l}+v_j^s w_l^s\frac{\partial^2 F_{s_k^i}}{\partial s_m^j\,\partial s_n^l}\\
&\quad=-(\delta_{n,k-m}+\delta_{n,m-k}+\delta_{n,m+k})[(v^r\cdot k)P_k(w^r)_i+(w^r\cdot k)P_k(v^s)_i]\\
&\qquad+(\delta_{n,k-m}-\delta_{n,m-k}-\delta_{n,m+k})[(v^s\cdot k)P_k(w^s)_i+(w^s\cdot k)P_k(v^r)_i],
\end{aligned}$$

where $P_k(v)_i=v_i-(k_i/|k|^2)(v\cdot k)$. We also analyze the coefficients of $\partial_{\tilde{r}_k}$ and $\partial_{\tilde{s}_k}$

$$\begin{aligned}
&\text{coefficients of }\partial_{\tilde{r}_k}\\
&\quad=[\tilde{v}^r(w^s\cdot k)+\tilde{v}^s(w^r\cdot k)](\delta_{n,k-m}-\delta_{n,m-k}+\delta_{n,m+k})\\
&\qquad+[\tilde{w}^s(v^r\cdot k)+\tilde{w}^s(v^s\cdot k)](\delta_{m,k-n}-\delta_{m,n-k}+\delta_{m,n+k}),
\end{aligned}$$

and

$$\begin{aligned}
&\text{coefficients of }\partial_{\tilde{s}_k}\\
&\quad=-\tilde{w}^r(v^r\cdot k)(\delta_{n,k-m}+\delta_{n,m-k}+\delta_{n,m+k})\\
&\qquad+\tilde{w}^s(v^s\cdot k)(\delta_{n,k-m}-\delta_{n,m-k}-\delta_{n,m+k})\\
&\qquad-\tilde{v}^r(w^r\cdot k)(\delta_{m,k-n}+\delta_{m,n-k}+\delta_{m,n+k})\\
&\qquad+\tilde{v}^s(w^s\cdot k)(\delta_{m,k-n}-\delta_{m,n-k}-\delta_{m,n+k}).
\end{aligned}$$

Thus this lemma is true. ∎

We define the set $A(\mathcal{N})$ of all the indices $k \in \mathcal{K}_N$ such that the constant vector fields corresponding to k if $k \in \tilde{\mathcal{K}}$, $-k$ if $k \in -\tilde{\mathcal{K}}$, are in the Lie algebra generated by the vector fields $\{F_0\} \cup \mathcal{U}_k$, $k \in \mathcal{N}$. The following lemma studies the algebraic structure of $A(\mathcal{N})$.

Lemma 4.2. Let $\mathcal{N}$ be a subset of indices and define the set $A(\mathcal{N})$ as above.

(i) If $m \in A(\mathcal{N})$, then also $-m \in A(\mathcal{N})$,

(ii) If $m, n \in A(\mathcal{N})$, $m+n \in \mathcal{K}_N$, m and n are linearly independent and $|m| \neq |n|$, then $m+n \in A(\mathcal{N})$,

(iii) If $|m| = |n|$, $V \in \mathcal{U}_m$, $W \in \mathcal{U}_n$, then the Lie bracket $[[F_0, V], W]$ span the four dimensional subspace of $\mathcal{U}_{m+n}$.

Proof. (i) follows simply from the property that $u_{-k} = \bar{u}_k$ and $\theta_{-k} = \bar{\theta}_k$. (ii) It is enough to prove that if $m, n \in A(\mathcal{N}) \cap \tilde{\mathcal{K}}$ and $k = m+n \in \tilde{\mathcal{K}}$, then $k \in A(\mathcal{N})$. Let

$$V^r = \sum_{i=1}^{3} v_i \frac{\partial}{\partial r_m^i} + \tilde{v} \frac{\partial}{\partial \tilde{r}_m}, \qquad V^s = \sum_{i=1}^{3} v_i \frac{\partial}{\partial s_m^i} + \tilde{v} \frac{\partial}{\partial \tilde{s}_m},$$
$$W^r = \sum_{i=1}^{3} w_i \frac{\partial}{\partial r_n^i} + \tilde{w} \frac{\partial}{\partial \tilde{r}_n}, \qquad W^s = \sum_{i=1}^{3} w_i \frac{\partial}{\partial s_n^i} + \tilde{w} \frac{\partial}{\partial \tilde{s}_n},$$

where $v \cdot m = w \cdot n = 0$. By the previous lemma, we have

$$\begin{aligned}[[F_0, V^r], W^s] + [[F_0, V^s], W^r] = {} & 2((v \cdot k) P_k(w) + (w \cdot k) P_k(v)) \cdot \frac{\partial}{\partial r_k} \\ & + 2((w \cdot k)\tilde{v} + (v \cdot k)\tilde{w}) \frac{\partial}{\partial \tilde{r}_k}\end{aligned}$$

and

$$\begin{aligned}[[F_0, V^r], W^r] - [[F_0, V^s], W^s] = {} & 2((v \cdot k) P_k(w) + (w \cdot k) P_k(v)) \cdot \frac{\partial}{\partial s_k} \\ & + 2((w \cdot k)\tilde{v} + (v \cdot k)\tilde{w}) \frac{\partial}{\partial \tilde{s}_k}\end{aligned}$$

Then let $H, I \in \mathbb{R}^3$ such that $\{k, H, I\}$ is a basis of $\mathbb{R}^3$, where $H, I \in \mathbb{R}^3$ span$\{x \in \mathbb{R}^3 | x \cdot k = 0\}$ and m, n are in the the spanning space of k, H.

By the assumption on m, n, it is always possible to choose the coefficient a_1, b_1, c_1, a_2, b_2, c_2 such that if

$$v = a_1 k + b_1 H + c_1 I, \qquad w = a_2 k + b_2 H + c_2 I,$$

then $2((v\cdot k)P_k w + (w\cdot k)P_k(v))$ can be any vector in the spanning space of H, I and $w\cdot m = v\cdot n = 0$. Then, obviously, since we can freely choose $\tilde{v}$ and $\tilde{w}$, $2((w\cdot k)\tilde{v} + (v\cdot k)\tilde{w})$ can be any number. Therefore, $\mathcal{U}_k$ is contained in the Lie algebra generated by the vector fields $\{F_0\}\cup\mathcal{U}_p$, $p\in\mathcal{N}$.
(iii) The proof follows from the similar argument in (ii). ∎

If $A(\mathcal{N}) = \mathcal{K}_N$ is satisfied, i.e., Hörmander's condition holds, then the transition semigroup generated by (4.1) and (4.2) is strongly Feller, i.e., the transition probability is regular.

Lemma 4.3. If $\mathcal{N}$ contains the three indices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, then the transition probability densities of the solution process are regular.

Proof. By iteratively using the previous lemma, we can show $A(\mathcal{N}) = \mathcal{K}_N$. For the details, see ref. 13. ∎

5. RECURRENCE OF NEIGHBORHOODS OF THE ORIGIN IN THE CASE OF GALERKIN TRUNCATION

In this section, we follow closely ref. 5. In the following, $|\cdot|_{L^2}$, $|\cdot|_{\tilde{L}^2}$, and $|\cdot|_{\mathbb{L}^2}$ are denoted by $\|\cdot\|$ for simplicity. Here we consider the Galerkin truncation of three-dimensional Boussinesq equations as stated in the introduction with a degenerately stochastic forcing In the following if the level of approximation play no explicit role, then we do not express the level of the approximations. We wish to show that starting from any position, the dynamics enters any neighborhood of the origin infinitely often.

We begin with two auxiliary lemmas and set all of the constants $\nu = \kappa = \sigma = 1$ for simplicity. For the general constant, we can obtain the estimates as in the same way of the energy estimates. We define

$$\mathcal{B}(c) = \left\{ g = (u,\theta)^T \in \mathbb{L}^2(\mathbb{T}^3) : \|g\| = \left(\int_{\mathbb{T}^3} |g(x)|^2\,dx\right)^{1/2} \leqslant c \right\}.$$

Lemma 5.1. Let $\mathcal{B}_0 = \mathcal{B}(C_0)$ and $\mathcal{B}_1 = \mathcal{B}(C_1)$ be two arbitrary balls about the origin and h be some positive constant. Then there exists a $T_0 = T_0(C_0, C_1) > 0$ so that for any $T \geqslant T_0$ there is a p^* with

$$\inf_{(u_0,\theta_0)\in\mathcal{B}_0} \mathbb{P}_{u_0,\theta_0}\left\{\begin{pmatrix} u(t) \\ \theta(t)\end{pmatrix} \in \mathcal{B}_1 \text{ for all } t\in[T, T+h]\right\} \geqslant p^* > 0.$$

Proof. Define $v(t)=u(t)-\hat{f}(t)$ and $\Theta(t)=\theta(t)-\hat{g}(t)$, where $\hat{f}(t)=W_u(t)-W_u(0)$ and $\hat{g}(t)=W_\theta(t)-W_\theta(0)$. We see that $v(t)$ and $\Theta(t)$ satisfy

$$\frac{\partial v}{\partial t}=\Delta v-\mathcal{P}_N(u\cdot\nabla)(v+\hat{f})+\Delta\hat{f}-\begin{pmatrix}0\\0\\\Theta+\hat{g}\end{pmatrix}, \tag{5.1}$$

$$\frac{\partial\Theta}{\partial t}=\Delta\Theta-\mathcal{P}_N(u\cdot\nabla)(\Theta+\hat{g})+\Delta\hat{g}. \tag{5.2}$$

Taking L^2 inner product of (5.1) and (5.2) with v and Θ, respectively, and using the fact that $\nabla u=\nabla v+\nabla\hat{f}$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 &\leqslant -\|\nabla v\|^2+C\|\nabla u\|\|\Delta\hat{f}\|\|v\|+\|\Delta\hat{f}\|\|v\|+\|\Theta\|\|v\|+\|\hat{g}\|\|v\|\\ &\leqslant -\frac{5}{8}\|\nabla v\|^2+C(\|v\|^2+\|\nabla\hat{f}\|^2+1)\|\Delta\hat{f}\|^2+4\|\Theta\|^2+4\|\hat{g}\|^2\end{aligned}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Theta\|^2 &\leqslant -\|\nabla\Theta\|^2+C\|\nabla u\|\|\Delta\hat{g}\|\|\Theta\|+\|\Delta\hat{g}\|\|\Theta\|\\ &\leqslant -\frac{1}{2}\|\nabla\Theta\|^2+C(\|\Theta\|^2+\|\nabla\hat{f}\|^2+1)\|\Delta\hat{g}\|^2+\frac{1}{128}\|\nabla v\|^2.\end{aligned}$$

In the above two inequalities, we use

$$\begin{aligned}C\|\nabla u\|\|\Delta\hat{f}\|\|v\| &\leqslant C\|\nabla v\|\|v\|\|\Delta\hat{f}\|+C\|v\|\|\nabla\hat{f}\|\|\Delta\hat{f}\|\\ &\leqslant \frac{1}{8}\|\nabla v\|^2+\|v\|^2\|\Delta\hat{f}\|^2+\frac{1}{8}\|v\|^2+C\|\nabla\hat{f}\|^2\|\Delta\hat{f}\|^2,\\ C\|\Delta u\|\|\Delta\hat{g}\|\|\Theta\| &\leqslant C\|\nabla v\|\|\Delta\hat{g}\|\|\Theta\|+C\|\nabla\hat{f}\|\|\Delta\hat{g}\|\|\Theta\|\\ &\leqslant \frac{1}{128}\|\nabla v\|^2+C\|\Theta\|^2\|\Delta\hat{g}\|^2+\frac{1}{2}\|\Theta\|^2+C\|\nabla\hat{f}\|^2\|\Delta\hat{g}\|^2\end{aligned}$$

and

$$\|\Theta\|\|v\|\leqslant\frac{1}{16}\|v\|^2+4\|\Theta\|^2.$$

Multiplying 16 on the differential inequality on Θ and adding two differential inequalities on v and Θ, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\|v\|^2+8\|\Theta\|^2\right)$$
$$\leqslant -\left(\frac{1}{2}-C_3\|\Delta\hat{f}\|^2\right)\|v\|^2-\left(4-C_4\|\Delta\hat{g}\|^2\right)\|\Theta\|^2$$
$$+C\|\nabla\hat{f}\|^2\|\Delta\hat{f}\|^2+C\|\nabla\hat{f}\|^2\|\Delta\hat{g}\|^2+C\|\Delta\hat{f}\|^2+C\|\Delta\hat{g}\|^2.$$

Fix any $\delta>0$ and define for any $T>0$

$$\Omega'(\delta,T)=\left\{k\in C([0,T+h];\mathbb{L}^2(\mathbb{T}^3)):\sup_{s\in[0,T]}\|\Delta k(s)\|\leqslant\min\left(\delta,\frac{1}{4C_3},\frac{2}{C_4}\right)\right\}.$$

If $(\hat{f},\hat{g})\in\Omega'$, then there exists a constant C_5 so that

$$\frac{1}{2}\|v(t)\|^2+8\|\Theta(t)\|^2\leqslant\left(\frac{1}{2}\|v(0)\|^2+8\|\Theta(0)\|^2\right)e^{-(1/2)t}$$
$$+C_5\left(\min\left(\delta,\frac{1}{4C_3},\frac{2}{C_4}\right)^2+\min\left(\delta,\frac{1}{4C_3},\frac{2}{C_4}\right)^4\right).$$

Hence if $\|(u(0),\theta(0))\|<C_0$, then given any $C_1>0$ there exists a T and a δ such that $\|(v(T),\Theta(T))\|<(C_1/2)$. For sufficiently small δ, we assume that $\|(\hat{f}(t),\hat{g}(t))\|<(C_1/2)$ for $t\in[T,T+h]$ if $(f,g)\in\Omega'$. For appropriate T and δ, we have $\|(u(t),\theta(t))\|\leqslant C_1$ for $t\in[T,T+h]$. Since for any $T\in(0,\infty)$ and $\delta_0>0$, Ω' is an open set in the supremum topology, we know $\mathbb{P}\{\Omega'\}>0$. This completes the proof. ∎

Lemma 5.2. If $\|u_0\|^2+\|\theta_0\|^2>C^2$ a.s. for some constant C, with $C^2>\mathcal{E}_0^u+\mathcal{E}_0^\theta$, where $\mathcal{E}_0^u$, $\mathcal{E}_0^\theta$ are defined in the Section 1, then

$$\mathbb{P}\{\tau_{\mathcal{C}}(u_0,\theta_0)\geqslant t\}\leqslant\frac{\mathbb{E}\{\|u_0\|^2+\|\theta_0\|^2\}}{C^2}\exp(-2\delta t),$$

where $\delta=1-(\mathcal{E}_0^u+\mathcal{E}_0^\theta)/C^2$, $\mathcal{C}=\mathcal{B}(C)$, and

$$\tau_{\mathcal{C}}((u_0,\theta_0))=\inf\{s>0:(u(s),\theta(s))\in\mathcal{C}\text{ given }(u(0),\theta(0))=(u_0,\theta_0)\}.$$

Proof. Define $Y(s,u)=e^{2\delta s}\|u(s)\|^2$ and $Z(s,\theta)=e^{2\delta s}\|\theta(s)\|^2$. Applying Ito's formula gives

$$dY(s)=[2\delta Y(s)-2e^{2\delta s}\|\nabla u(s)\|^2+e^{2\delta s}\langle\theta(s),u(s)\rangle_{L^2}+e^{2\delta s}\mathcal{E}_0^u]\,ds$$
$$+2e^{2\delta s}\langle u,dW_u\rangle_{L^2}$$

and

$$dZ(s)=[2\delta Z(s)-2e^{2\delta s}\|\nabla\theta(s)\|^2+e^{2\delta s}\mathcal{E}_0^\theta]ds+2e^{2\delta s}\langle\theta,\, dW_\theta\rangle_{L^2}.$$

Adding above two equalities and using Young's inequality gives us that

$$\begin{aligned} d(Y(s)+Z(s))\leqslant\ & [\mathcal{E}_0^u+\mathcal{E}_0^\theta-(1-\delta)(\|u(s)\|^2+\|\theta(s)\|^2)]e^{2\delta s}\,ds \\ & +2e^{2\delta s}\langle u,\, dW_u\rangle_{L^2}+2e^{2\delta s}\langle\theta,\, dW_\theta\rangle_{L^2}. \end{aligned}$$

Define $S_n=\inf\{s>0:\|u(s)\|^2+\|\theta(s)\|^2>n(\|u_0\|^2+\|\theta_0\|^2)\}$ for each $n>1$. Then fix any t and let $T=\tau_{\mathcal{C}}\wedge S_n\wedge t$. Integrating the previous equation up to the stopping time T, it follows that

$$\begin{aligned} \mathbb{E}(Y(T)+Z(T))\leqslant\ & \mathbb{E}(Y(0)+Z(0)) \\ & +(\mathcal{E}_0^u+\mathcal{E}_0^\theta)\mathbb{E}e^{2\delta T}\int_0^T\left(1-\frac{\|u(s)\|^2+\|\theta(s)\|^2}{C^2}\right)ds \\ & +2\mathbb{E}\int_0^T e^{2\delta s}(\langle u,\, dW_u\rangle_{L^2}+\langle\theta,\, dW_\theta\rangle_{L^2}). \end{aligned} \tag{5.3}$$

Optional stopping time lemma gives us that

$$\begin{aligned} & \mathbb{E}\int_0^T e^{2\delta s}(\langle u(x,s),\, dW_u(x,s)\rangle_{L^2}+\langle\theta(x,s),\, dW_\theta(x,s)\rangle_{L^2}) \\ & =\mathbb{E}\int_0^T e^{2\delta s}(\langle u(x,s\wedge S_n),\, dW_u(x,s)\rangle_{L^2}+\langle\theta(x,s\wedge S_n),\, dW_\theta(x,s)\rangle_{L^2}) \\ & =0. \end{aligned}$$

For $t<\tau_{\mathcal{C}}$, we have $\|u(s)\|^2+\|\theta(s)\|^2>C^2$. Hence we obtain $\mathbb{E}(Y(t\wedge\tau_{\mathcal{C}})+Z(t\wedge\tau_{\mathcal{C}}))\leqslant\mathbb{E}(Y(0)+Z(0))$ by $n\to\infty$. Since $\|u(s)\|^2+\|\theta(s)\|^2$ is finite and continuous in time,we have $S_n\to\infty$ as $n\to\infty$. Letting $n\to\infty$, we have $\mathbb{E}((Y+Z)(t\wedge\tau_{\mathcal{C}}))\leqslant\mathbb{E}((Y+Z)(0)))$. Then we have

$$\begin{aligned} \mathbb{E}(Y(0)+Z(0))\geqslant\ & \mathbb{E}\{Y(\tau_{\mathcal{C}})+Z(\tau_{\mathcal{C}})\,|\,t>\tau_{\mathcal{C}}\}\mathbb{P}\{t>\tau_{\mathcal{C}}\} \\ & +\mathbb{E}\{Y(t)+Z(t)\,|\,t\leqslant\tau_{\mathcal{C}}\}\mathbb{P}\{t\leqslant\tau_{\mathcal{C}}\} \\ \geqslant\ & C^2e^{2\delta t}\mathbb{P}\{t\leqslant\tau_{\mathcal{C}}\}. \end{aligned}$$

This completes the proof of lemma. ∎

Lemma 5.3. Fix an $h>0$ and an open neighborhood U_1 of the origin. Then given any initial condition $\begin{pmatrix} u_0 \\ \theta_0 \end{pmatrix}$,

$$\mathbb{P}_{u_0,\theta_0}\left\{\begin{pmatrix} u(nh) \\ \theta(nh) \end{pmatrix} \in U_1 \text{ for infinitely many } n\right\}=1.$$

Proof. Define C and $\mathcal{C}$ as in Lemma 5.2. Since U_1 is open, we can pick C_1 small enough $\mathcal{B}(C_1)\subset U_1$. Let T_0 be given by Lemma 5.1 when $\mathcal{B}(C_0)=\mathcal{C}$, where $\begin{pmatrix} u_0 \\ \theta_0 \end{pmatrix}\in\mathcal{B}(C_0)$. Now let T be the smallest integer multiple of h that is greater than (T_0+2h) and set $n^*=(T/h)$. Define $\begin{pmatrix} u(nh) \\ \theta(nh) \end{pmatrix}$ by v_n. By Lemma 5.1, there exists a $p^*>0$ so that

$$\mathbb{P}\left\{v_{n+n^*-1}\in U_1 \mid \begin{pmatrix} u(t) \\ \theta(t) \end{pmatrix}\in\mathcal{C} \text{ for some } t\in[(n-1)h,\, nh]\right\}\geqslant p^*.$$

Define the sequence of increasing integer stopping times τ_n by

$$\tau_0=\inf\left\{n\geqslant 1 : \begin{pmatrix} u(t) \\ \theta(t) \end{pmatrix}\in\mathcal{C} \text{ for some } t\in[(n-1)h,\, nh]\right\}$$

and for $k>0$

$$\tau_k=\inf\left\{n\geqslant \tau_{k-1}+(n^*+1) : \begin{pmatrix} u(t) \\ \theta(t) \end{pmatrix}\in\mathcal{C} \text{ for some } t\in[(n-1)h,\, nh]\right\}.$$

By Lemma 5.2, it is clear that each τ_k is almost surely finite. Let us define $\#_{U_1}(n)$ as the number of $k\in[0,n]$ so that $v_n\in U_1$. By Lemma 5.1, we have that for any n and M, with $M<n$,

$$\mathbb{P}\{\#_{U_1}(\tau_n+n^*)<M\}\leqslant(1-p^*)^{n-M}.$$

Hence we see that U_1 is visited infinitely often almost surely. ∎

Proof of Theorem 1.2. The proof is almost same as the proof of ref. 5. But for the completeness, we provide the proof. Let p_t be a transition density of the system (4.1) and (4.2). Fix $h>0$. Since $\int p_h(0,\, y)dy=1$, there exists a y_0 such that $p_h(0,\, y_0)>0$. By Lemma 4.3, there exists an open neighborhood of 0 denoted by A_1, an open neighborhood of y_0, denoted by A_2, and a positive constant δ_0 such that $p_h(x,\, y)>\delta_0$ if $x\in A_1$,

$y \in A_2$. Let m be the normalized Lebesgue measure on A_2. We claim that Markov chain $(u_n, \theta_n)^T$ by setting $(u_n, \theta_n)^T = (u(nh), \theta(nh))^T$ satisfies Harris' condition, i.e., we need to show that for any measurable $B \subset A_2$ such that $m(B) > 0$,

$$\mathbb{P}_{u_0,\theta_0}\{(u_n, \theta_n) \in B \text{ for infinitely many } n\} = 1.$$

Let t_n be the n-th time such that (u_n, θ_n) is in A_1. By Lemma 5.3, we see that t_n's are well defined and finite with probability 1. Define $\#_B(n)$ as in the proof of Lemma 5.3, to be the number of $k \in [0, n]$ such that $(u_k, \theta_k) \in B$. Since $B \subset A_2$, we have

$$\mathbb{P}_{u_0,\theta_0}\{(u_n, \theta_n) \in B | (u_{n-1}, \theta_{n-1}) \in A_1\} = \int_B p_h((u_{n-1}, \theta_{n-1}), y) dy \geqslant \delta_0 m(B).$$

We have $\delta_1 = \delta_0 m(B) > 0$ because $m(B) > 0$. Fix some positive M and n with $n > M$. Hence we have

$$\mathbb{P}\{\#_B(t_{n+1}) < M\} \leqslant (1 - \delta_1)^{n-M}.$$

If $n \to \infty$, $\mathbb{P}\{\#_B(t_{n+1}) < M\} \to 0$. Since M is arbitrary, B is visited infinitely many times almost surely. This completes the proof of Theorem 1.2. ∎

Remark. In the proof of Lemma 4.2, we use the peculiar property of the three dimension. For the Galerkin approximations of two-dimensional Boussinesq equations, we can show some modes (e.g. $(1, 0)$, $(1, 1)$) satisfy (ii) of Lemma 4.2 by direct calculations (see ref. 5). The recurrence of neighborhoods of the origin is still true for two-dimensional Galerkin truncation. Thus we can obtain the similar result with Theorem 1.2 (e.g. for the case that set of modes forced includes $(1, 0)$ and $(1, 1)$) for the two-dimensional Galerkin truncation of the Boussinesq system. It is also interesting to look for other cases(even minimal conditions) for which Theorem 1.2 holds.

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